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Excerpt
More information

1

Point sets and certain classes of sets

1.1 Points, sets and classes

We shall consider *sets* consisting of *elements* or *points*. The nature of the points will be left unspecified – examples are points in a Euclidean space, sequences of numbers, functions, elementary events, etc. Small letters will be used for points.

Sets are aggregates or collections of such points. Capital letters will be used for sets.

A set is defined by a property. That is, given a point, there is a criterion to decide whether it belongs to a given set, e.g. the set which is the open interval $(-1, 1)$ on the real line is defined by the property that it contains a point x if and only if $|x| < 1$.

A set may be written as $\{x : P(x)\}$ where $P(x)$ is the property defining the set; e.g. $\{x : |x| < 1\}$ is the above set consisting of all points x for which $|x| < 1$, i.e. $(-1, 1)$.

In any given situation, all the points considered will belong to a fixed set called the *whole space* and usually denoted by X. This assumption avoids some difficulties which arise in the logical foundations of set theory.

Classes or *collections* of sets are just aggregates whose elements themselves are sets, e.g. the class of all intervals of the real line, the class of all circles in the plane whose centers are at the origin, and so on. Script capitals will be used for classes of sets.

Collections of *classes* are similarly defined to be aggregates whose elements are classes. Similarly, higher logical structures may be defined.

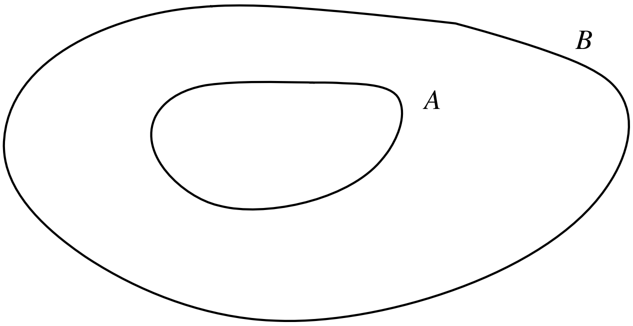
Note that a class of sets, or a collection of classes, is itself a *set*. The words "class of sets" are used simply to emphasize that the elements are themselves sets (in some fixed whole space X).

1.2 Notation and set operations

$\in$ — $x \in A$ means that the point x is an element of the set A. This symbol can also be used between sets and classes, e.g. $A \in \mathcal{A}$ means the set A is a member of the class $\mathcal{A}$. The symbol $\in$ must be used between entities of *different* logical type, e.g. *point* $\in$ *set*, *set* $\in$ *class of sets*.

$\notin$ — The opposite of $\in$, $x \notin A$ means that the point x is not an element of the set A.

$\subset$ — $A \subset B$ (or $B \supset A$) means that the set A is a subset of B. That is, every element of A is also an element of B, or $x \in A \Rightarrow x \in B$ (using "$\Rightarrow$" for "implies"). Diagrammatically, one may think of sets in the plane:



$A \subset B$.

The symbol $\subset$ is used between entities of the *same* logical type such as *sets* ($A \subset B$), or *classes of sets* ($\mathcal{A} \subset \mathcal{B}$ meaning every set in the class $\mathcal{A}$ is also in the class $\mathcal{B}$. $\mathcal{A}$ is a *subclass* of $\mathcal{B}$).

Examples

$$A = \{x : |x| \le 1/2\} = [-1/2, 1/2],$$
$$B = \{x : |x| < 1\} = (-1, 1),$$
$$(A \subset B),$$
$$\mathcal{A} = \text{ class of all intervals of the form } (n, n+1) \text{ for } n = 1, 2, 3, \ldots,$$
$$\mathcal{B} = \text{ class of all intervals,}$$
$$(\mathcal{A} \subset \mathcal{B}).$$

Note that $A \subset A$, i.e. the symbol $\subset$ does not preclude equality.

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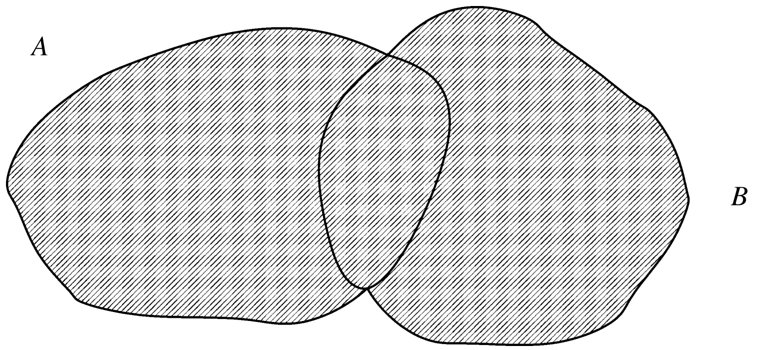
$=$ *Equals* If $A \subset B$ and $B \subset A$ we write $A = B$. That is A and B consist of the same points.

$\emptyset$ The *empty set*, i.e. the set with no points in it. Note by definition $\emptyset \subset A$ for any set A. Also if X denotes the whole space, $A \subset X$ for any set A.

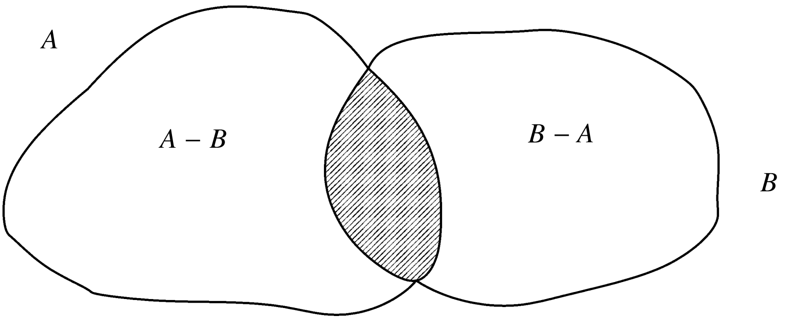
$\cup$ The *union* (sum) of two sets A and B, written $A \cup B$ is the set of all points in *either* A or B (or both). That is

$$A \cup B = \{x : x \in A \text{ or } x \in B \text{ or both}\}.$$



$A \cup B$ is the entire shaded area.

$\cap$ The *intersection* of two sets A and B, written $A \cap B$ is the set of all points in *both* A and B.



$A \cap B$ (shaded area)
$A - B$, $B - A$ (unshaded areas).

Two sets A, B with no points in common ($A \cap B = \emptyset$) are said to be *disjoint*. A *class* of sets is called disjoint if each pair of its members is disjoint.

Sometimes AB is written for $A \cap B$, and $A + B$ for $A \cup B$ (though $A + B$ is sometimes reserved for the case when $A \cap B = \emptyset$).

The *difference* of two sets. $A - B$ is the set of all points of A which are not in B, i.e. $\{x : x \in A \text{ and } x \notin B\}$.

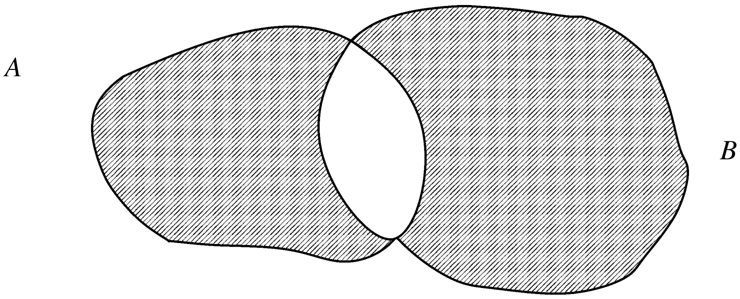
If $B \subset A$, $A - B$ is called a *proper* difference. Note the need for care with algebraic laws, e.g. in general

$$(A - B) \cup C \neq (A \cup C) - B.$$

The *complement* A^c of a set A consists of all points of the space X which are not in A, i.e. $A^c = X - A$.

The *symmetric difference* $A \Delta B$ of A and B is the set of all points which are in either A or B but not both, i.e.

$$A \Delta B = (A - B) \cup (B - A).$$



$A \Delta B$ = shaded area.

Unions and intersections of arbitrary numbers of sets:

If A_γ is a set for each γ in some index set Γ, $\cup_{\gamma \in \Gamma} A_\gamma$ is the set of all points which are members of at least one of the A_γ.

$$\cup_{\gamma \in \Gamma} A_\gamma = \{x : x \in A_\gamma \text{ for } \textit{some } \gamma \in \Gamma\}.$$
$$\cap_{\gamma \in \Gamma} A_\gamma = \{x : x \in A_\gamma \text{ for } \textit{all } \gamma \in \Gamma\}.$$

If Γ is, for example, the set of positive integers, we write $\cup_{n=1}^\infty$ for $\cup_{n \in \Gamma}$, etc. For example, $\cup_{n=1}^\infty [n, n+1] = [1, \infty)$, where [ ] denotes a closed interval and [ ) semiclosed, etc., and $\cap_{n=1}^\infty [0, \frac{1}{n}) = \{0\}$, the set consisting of the single point 0 only. Also $\cap_{n=1}^\infty (0, \frac{1}{n}) = \emptyset$.

The set operations $\cup, \cap, -, \Delta$ have been defined for sets but of course they apply also to *classes* of sets; e.g. $\mathcal{A} \cap \mathcal{B} = \{A : A \in \mathcal{A} \text{ and } A \in \mathcal{B}\}$ is the class of all those sets which are members of *both* the classes $\mathcal{A}$ and $\mathcal{B}$. (Care should be taken – cf. Ex. 1.3!)

1.3 Elementary set equalities

To prove a set equality $A = B$, it is necessary by definition, to show that $A \subset B$ and $B \subset A$ (i.e. that A and B consist of the same points). Thus we first take any point $x \in A$ and show $x \in B$; then we take any point $y \in B$ and show $y \in A$. The following result summarizes a number of simple set equalities.

Theorem 1.3.1 *For any sets $A, B, \ldots,$*

(i) $A \cup B = B \cup A, \quad A \cap B = B \cap A$ *(commutative laws)*
(ii) $(A \cup B) \cup C = A \cup (B \cup C), \quad (A \cap B) \cap C = A \cap (B \cap C)$ *(associative laws)*
(iii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ *(distributive law)*
(iv) $E \cap \emptyset = \emptyset, \quad E \cup \emptyset = E$
(v) $E \cap X = E, \quad E \cup X = X$
(vi) *If $E \subset F$ then $E \cap F = E$ and conversely*
(vii) $E - F = E \cap F^c$ *for all E, F*
(viii) $E - (F \cup G) = (E - F) \cap (E - G), \quad E - (F \cap G) = (E - F) \cup (E - G)$
(ix) $(\cup_{\gamma\in\Gamma} A_\gamma)^c = \cap_{\gamma\in\Gamma} A_\gamma^c, \quad (\cap_{\gamma\in\Gamma} A_\gamma)^c = \cup_{\gamma\in\Gamma} A_\gamma^c.$

These are easily verified and we prove just two ((iii) and (ix)) by way of illustration. As already noted, the symbol $\Rightarrow$ is used to denote "implies", "LHS" for "left hand side", etc.

Proof of (iii)

$$\begin{aligned} x \in \text{LHS} &\Rightarrow x \in A \text{ and } x \in B \cup C \\ &\Rightarrow x \in A, \text{ and } x \in B \text{ or } x \in C \\ &\Rightarrow x \in A \text{ and } B, \text{ or } x \in A \text{ and } C \\ &\Rightarrow x \in A \cap B \text{ or } x \in A \cap C \\ &\Rightarrow x \in \text{ RHS}. \end{aligned}$$

Thus LHS $\subset$ RHS. Similarly RHS $\subset$ LHS, showing equality. Both inclusions may actually be obtained together by noting that each statement not only implies the next, but is equivalent to it, i.e. we may write "$\Leftrightarrow$" ("implies and is implied by" or "is equivalent to") instead of the one way implication $\Rightarrow$. From this we obtain $x \in$ LHS $\Leftrightarrow x \in$ RHS, giving inclusion both ways and hence equality. □

Proof of (ix) The same style of proof as above may be used here, of course. Instead it may be set out in a slightly different way using the notation $\{x : P(x)\}$ defining a set by its property P. For the first equality

$$\begin{aligned}(\cup A_\gamma)^c &= \{x : x \notin \cup A_\gamma\} \\ &= \{x : x \notin A_\gamma \text{ for any } \gamma\} \\ &= \{x : x \in A_\gamma^c, \text{ all } \gamma\} \\ &= \cap A_\gamma^c.\end{aligned}$$

The second equality follows similarly or by replacing A_γ by A_γ^c in the first to obtain $\cap A_\gamma = (\cup A_\gamma^c)^c$ and hence $(\cap A_\gamma)^c = \cup A_\gamma^c$. □

The equality (ii) may, of course, be extended to show that the terms of a union may be grouped in any way and taken in any order, and similarly for the terms of an intersection. (This is not always true for a mixture of unions and intersections, e.g. $A \cap (B \cup C) \neq (A \cap B) \cup C$, in general, but rather laws such as (iii) hold.)

(viii) and (ix) are sometimes known as "De Morgan laws". (ix) states that the "complement of a union is the intersection of the complements", and the "complement of an intersection is the union of the complements". (viii) is essentially just a simpler case of this with complements taken "relative to a fixed set E". In fact (viii) follows from (ix) (and (vii)) e.g. by noting that

$$\begin{aligned}E - (F \cup G) &= E \cap (F \cup G)^c = E \cap F^c \cap G^c = (E \cap F^c) \cap (E \cap G^c) \\ &= (E - F) \cap (E - G).\end{aligned}$$

1.4 Limits of sequences of sets

Let $\{E_n : n = 1, 2, \ldots\}$ be a sequence of subsets of X.

$\overline{\lim} E_n$ (the *upper limit* of $\{E_n\}$) is the set of all points x which belong to E_n for infinitely many values of n. That is, given *any* m, there is some $n \geq m$ with $x \in E_n$ (i.e. we may say $x \in E_n$ "infinitely often" or "*for arbitrarily* large values of n").

$\underline{\lim} E_n$ (the *lower limit* of $\{E_n\}$) is the set of all points x such that x belongs to *all but a finite number* of E_n. That is $x \in E_n$ for all $n \geq n_0$

where n_0 is some integer (which will usually be different for different x). Equivalently, we say $x \in E_n$ "*for all sufficiently* large values of n".

Theorem 1.4.1 *For any sequence $\{E_n\}$ of sets*

(i) $\overline{\lim} E_n = \cap_{n=1}^{\infty} \cup_{m=n}^{\infty} E_m$
(ii) $\underline{\lim} E_n = \cup_{n=1}^{\infty} \cap_{m=n}^{\infty} E_m$.

Proof To show (ii):
$x \in \underline{\lim} E_n \Rightarrow x \in E_n$ for all $n \geq$ some n_0, and thus $x \in \cap_{m=n_0}^{\infty} E_m$. Hence $x \in \cup_{n=1}^{\infty} (\cap_{m=n}^{\infty} E_m)$.

Conversely if $x \in$ RHS of (ii) then, for some n_0, $x \in \cap_{m=n_0}^{\infty} E_m$, and hence $x \in E_m$ for all $m \geq n_0$. Thus $x \in \underline{\lim} E_n$ as required. Similarly for the proof of (i). □

A sequence $\{E_n\}$ is called *convergent* if $\underline{\lim} E_n = \overline{\lim} E_n$ and we then write $\lim E_n$ for this set. Since clearly $\underline{\lim} E_n \subset \overline{\lim} E_n$, to show a sequence $\{E_n\}$ is convergent it need only be shown that $\overline{\lim} E_n \subset \underline{\lim} E_n$.

A sequence $\{E_n\}$ is called *monotone increasing (decreasing)* if $E_n \subset E_{n+1}$ ($E_n \supset E_{n+1}$) for all n. These are conveniently written respectively as $E_n \uparrow$, $E_n \downarrow$.

Theorem 1.4.2 *A monotone increasing (decreasing) sequence $\{E_n\}$ is convergent and* $\lim E_n = \cup_{n=1}^{\infty} E_n$ $(\cap_{n=1}^{\infty} E_n)$.

Proof If $E_n \uparrow$ (i.e. monotone increasing),

$$\overline{\lim} E_n = \cap_{n=1}^{\infty} (\cup_{m=n}^{\infty} E_m) = \cap_{n=1}^{\infty} (\cup_{m=1}^{\infty} E_m)$$

since $\cup_{m=1}^{\infty} E_m = \cup_{m=n}^{\infty} E_m$ $(E_m \uparrow)$. But $\cup_{m=1}^{\infty} E_m$ does not depend on n and thus

$$\overline{\lim} E_n = \cup_{m=1}^{\infty} E_m.$$

But also $\underline{\lim} E_n = \cup_{n=1}^{\infty} \cap_{m=n}^{\infty} E_m = \cup_{n=1}^{\infty} E_n$ since $\cap_{m=n}^{\infty} E_m = E_n$.

Hence $\overline{\lim} E_n = \cup_{n=1}^{\infty} E_n = \underline{\lim} E_n$ as required. Similarly for the case $E_n \downarrow$ (i.e. monotone decreasing). □

1.5 Indicator (characteristic) functions

If E is a set, its *indicator* (or *characteristic*) function $\chi_E(x)$ is defined by

$$\chi_E(x) = 1 \text{ for } x \in E$$
$$= 0 \text{ for } x \notin E.$$

This function determines E since E is the set of points x for which the value of the function is one, i.e. $E = \{x : \chi_E(x) = 1\}$.

Simple properties:

$$\begin{aligned}
\chi_E(x) &\le \chi_F(x), \text{ all } x \Leftrightarrow E \subset F\\
\chi_E(x) &= \chi_F(x), \text{ all } x \Leftrightarrow E = F\\
\chi_\emptyset(x) &\equiv 0, \quad \chi_X(x) \equiv 1\\
\chi_{E^c}(x) &= 1 - \chi_E(x), \text{ all } x\\
\chi_{\cap_1^n E_i}(x) &= \textstyle\prod_1^n \chi_{E_i}(x).
\end{aligned}$$

If E_i are disjoint,

$$\chi_{\cup_1^n E_i}(x) = \textstyle\sum_1^n \chi_{E_i}(x).$$

1.6 Rings, semirings, and fields

One of the most basic concepts in measure theory is that of a *ring* of sets.

Specifically a *ring* is a nonempty class $\mathcal{R}$ of subsets of the space X such that if $E \in \mathcal{R}$, $F \in \mathcal{R}$, then $E \cup F \in \mathcal{R}$ and $E - F \in \mathcal{R}$.

Put in another way a ring is a nonempty class $\mathcal{R}$ which is *closed* under the formation of unions and differences (of any two of its sets).[1] The following result summarizes some simple properties of rings.

Theorem 1.6.1 *Every ring contains the empty set $\emptyset$. A ring is closed under the formation of*

(i) symmetric differences and intersections

(ii) finite *unions and* finite *intersections (i.e. if $E_1, E_2, \ldots, E_n \in \mathcal{R}$, then $\cup_1^n E_i \in \mathcal{R}$ and $\cap_1^n E_i \in \mathcal{R}$).*

Proof Since $\mathcal{R}$ is nonempty it contains some set E and hence $\emptyset = E - E \in \mathcal{R}$. If $E, F \in \mathcal{R}$, then

$$\begin{aligned}
E\Delta F &= (E - F) \cup (F - E) \in \mathcal{R} \ \ (\text{since } (E - F), \ \ (F - E) \in \mathcal{R})\\
E \cap F &= (E \cup F) - (E\Delta F) \in \mathcal{R} \ \ (\text{since } E \cup F, \ \ E\Delta F \in \mathcal{R}).
\end{aligned}$$

Thus (i) follows. (ii) follows by induction since e.g. $\cup_1^n E_i = \left(\cup_1^{n-1} E_i\right) \cup E_n$. (See also Footnote 1.) □

The next result gives an alternative criterion for a class to be a ring.

[1] Whenever we say a class is "closed under unions" (or "closed under intersections") it is meant that the union (or intersection) of any *two* (and hence, by induction as above, any *finite* number of) members of the class, belongs to the class. If countable unions or intersections are involved, this will be expressly stated.

Theorem 1.6.2 *Let $\mathcal{R}$ be a nonempty class of sets which is closed under formation of either*

(i) unions and proper differences or
(ii) intersections, proper differences and disjoint unions.

Then $\mathcal{R}$ is a ring.

Proof Suppose (i) holds. Then if $E, F \in \mathcal{R}$, $E - F = (E \cup F) - F \in \mathcal{R}$ since this is a proper difference of sets of $\mathcal{R}$. Hence $\mathcal{R}$ is a ring.

If now (ii) holds and $E, F \in \mathcal{R}$, then

$$E \cup F \;=\; (E - (E \cap F)) \cup F.$$

This expresses $E \cup F$ as a disjoint union of sets of $\mathcal{R}$. Hence $E \cup F \in \mathcal{R}$. Thus (i) holds so that $\mathcal{R}$ is a ring. □

Trivial examples of rings are

(i) the class $\{\emptyset\}$ consisting of the empty set only
(ii) the class of all subsets of X.

More useful rings will be considered later.

The next result is a useful lemma which shows how a union of a sequence of sets of a ring $\mathcal{R}$ may be expressed either as a union of an increasing sequence or a disjoint sequence, of sets of $\mathcal{R}$.

Lemma 1.6.3 *Let $\{E_n\}$ be a sequence of sets of a ring $\mathcal{R}$, and $E = \cup_1^\infty E_n$ (E is not necessarily in $\mathcal{R}$). Then*

(i) $E = \cup_1^\infty F_n = \lim F_n$ where $F_n = \cup_{i=1}^n E_i$ are increasing sets in $\mathcal{R}$
(ii) $E = \cup_1^\infty G_n$ where G_n are disjoint sets of $\mathcal{R}$, such that $G_n \subset E_n$.

Proof (i) is immediate.

(ii) follows from (i) by writing $G_1 = E_1$ and $G_n = F_n - F_{n-1}$ $(\subset E_n)$, for $n > 1$. Clearly the G_n are in $\mathcal{R}$, are disjoint since F_n are increasing, and $\cup_1^\infty F_n = \cup_1^\infty G_n$, completing the proof. □

Fields. A *field* (or *algebra*) is a nonempty class $\mathcal{F}$ of subsets of X such that if $E \in \mathcal{F}$, then $E^c \in \mathcal{F}$ and if $E, F \in \mathcal{F}$ then $E \cup F \in \mathcal{F}$. That is, a field is closed under the formation of unions and complements.

Theorem 1.6.4 *A field is a ring of which the whole space X is a member, and conversely.*

Proof Let $\mathcal{F}$ be a field, and let $E \in \mathcal{F}$. Then $E^c \in \mathcal{F}$ and hence $X = E \cup E^c \in \mathcal{F}$.

Further, if $E \in \mathcal{F}$, $F \in \mathcal{F}$, then

$$E - F = E \cap F^c = (E^c \cup F)^c \in \mathcal{F}$$

(using the field axioms). Thus $\mathcal{F}$ is a ring and contains X.

Conversely, if $\mathcal{F}$ is a ring containing X and $E \in \mathcal{F}$, we have $E^c = X - E \in \mathcal{F}$. Thus $\mathcal{F}$ is a field. □

The next lemma shows that the intersection of an arbitrary collection of rings (or fields) is a ring (or field). In fact such a result applies much more widely to many (but not all!) classes defined by very general closure properties (and exactly the same method of proof may be used. This will be seen later in further important cases).

Lemma 1.6.5 *Let $\mathcal{R}_\gamma$ be a ring, for each γ in an arbitrary index set Γ (which may be finite, countable or uncountable). Let $\mathcal{R} = \cap\{\mathcal{R}_\gamma : \gamma \in \Gamma\}$ i.e. $\mathcal{R}$ is the class of all sets E belonging to* every *$\mathcal{R}_\gamma$ for $\gamma \in \Gamma$. Then $\mathcal{R}$ is a ring.*

Proof If $E, F \in \mathcal{R}$ then $E, F \in \mathcal{R}_\gamma$ for every $\gamma \in \Gamma$. Since $\mathcal{R}_\gamma$ is a ring it follows that $E-F$ and $E \cup F$ belong to each $\mathcal{R}_\gamma$ and hence $E-F \in \mathcal{R}$, $E \cup F \in \mathcal{R}$.

Finally the empty set $\emptyset$ belongs to every $\mathcal{R}_\gamma$ and hence to $\mathcal{R}$ which is therefore a nonempty class, and hence is a ring. □

A useful class of sets which is less restrictive than a ring is a *semiring*. Specifically, a semiring is a nonempty class $\mathcal{P}$ of sets such that

(i) If $E \in \mathcal{P}, F \in \mathcal{P}$, then $E \cap F \in \mathcal{P}$,

(ii) If $E \in \mathcal{P}, F \in \mathcal{P}$, then $E - F = \cup_1^n E_i$, where n is some positive integer and $E_1, E_2, \ldots, E_n$ are disjoint sets of $\mathcal{P}$.

Clearly the empty set $\emptyset$ belongs to any semiring $\mathcal{P}$ since there is some set $E \in \mathcal{P}$ and hence by (ii) $\emptyset = E - E = \cup_i^n E_i$ for some n, $E_i \in \mathcal{P}$. But this implies that each E_i is empty so that $\emptyset = E_i \in \mathcal{P}$.

A ring is clearly a semiring. In the real line, the class of all semiclosed intervals of the form $a < x \leq b$ $((a, b])$ is a semiring which is not a ring. However, the class of all finite unions of semiclosed intervals is a ring – as will be seen in the next section.