1

Function spaces

In this chapter we introduce our (mostly standard) notation and recall some basic facts about the function spaces we use most often in this book. Less standard material is covered in Section 1.7.1 (Sobolev spaces on $\mathbb{T}^3$) and Section 1.9 (Bochner spaces).

1.1 Domain of the flow

We consider almost exclusively flows in a three-dimensional domain. We focus mainly on two non-physical domains without boundaries:

- the whole space $\mathbb{R}^3$ and
- the three-dimensional torus $\mathbb{T}^3$.

As discussed in the Introduction, we will supplement the problem with appropriate additional conditions: on the whole space we will require some decay at infinity (for example, $u \in L^2(\mathbb{R}^3)$, which corresponds to finite kinetic energy), while on the torus it is convenient to take solutions with zero average, $\int_{\mathbb{T}^3} u = 0$. We concentrate in this chapter on defining function spaces on these domains, taking these constraints into account.

Throughout the book we also state the corresponding results for the physical case

- $\Omega \subset \mathbb{R}^3$ is a simply-connected bounded open set with a smooth boundary, which we call a 'smooth bounded domain' for short; on this kind of domain we always impose[1] the Dirichlet boundary condition $u|_{\partial\Omega} = 0$.

[1] "It turns out – although it is not at all self-evident – that in all circumstances where it has been experimentally checked, *the velocity of a fluid is exactly zero at the surface of a solid*." (Feynman, Leighton, & Sands, 1970, their italics)

We highlight when results in this chapter require bounded domains or the absence of boundaries; an analysis on the torus can take advantage of both of these simplifications, which is why many of the results in the book (particularly in Part I) are proved in this case: the exposition is simplified but the essential difficulties remain.

Functions defined on the torus $\mathbb{T}^3 = (\mathbb{R}/2\pi\mathbb{Z})^3$ can be realised as periodic functions defined on all of $\mathbb{R}^3$, i.e.

$$u(x + 2\pi k) = u(x) \qquad \text{for all} \quad k \in \mathbb{Z}^3; \tag{1.1}$$

it is also convenient at times to identify u with its restriction to the fundamental domain $[0, 2\pi)^3$.

We will usually not distinguish in notation between spaces of scalar and vector-valued functions; when some ambiguity could arise we will use a notation like $[L^p]^n$ (this example denotes n-component vector-valued functions with each component in L^p) and in Chapter 2, in which the distinction between scalar and vector functions is significant, we will define $\mathbb{L}^p := [L^p]^3$.

1.2 Derivatives

We will use the notation ∂_j for the partial derivative corresponding to the jth coordinate. Two combinations of first derivatives will be particularly significant in what follows: if $u\colon \mathbb{R}^3 \to \mathbb{R}^3$ is a vector field then we can define the divergence of u as

$$\operatorname{div} u = \nabla \cdot u = \partial_i u_i$$

(summing, as ever, over repeated indices) and the curl of u as

$$\operatorname{curl} u = \nabla \wedge u = \begin{vmatrix} \underline{e}_1 & \underline{e}_2 & \underline{e}_3 \\ \partial_1 & \partial_2 & \partial_3 \\ u_1 & u_2 & u_3 \end{vmatrix},$$

where $\underline{e}_1$, $\underline{e}_2$, and $\underline{e}_3$ are the unit vectors along the coordinate axes. For calculations it is sometimes convenient to express the curl in the form

$$(\operatorname{curl} u)_i = \epsilon_{ijk}\partial_j u_k,$$

where ϵ_{ijk} is the Levi–Civita tensor defined by

$$\epsilon_{ijk} = \begin{cases} 1 & ijk \text{ is an even permutation of } 123 \\ -1 & ijk \text{ is an odd permutation of } 123 \\ 0 & \text{otherwise.} \end{cases} \tag{1.2}$$

We also define

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases} \tag{1.3}$$

The equality

$$\epsilon_{ijk}\epsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}, \tag{1.4}$$

which is useful for proving vector identities such as

$$\operatorname{curl}\operatorname{curl} u = \nabla(\nabla \cdot u) - \Delta u$$

(see Exercise 2.2), can be easily (if painfully) checked by hand.

For higher-order derivatives we will employ multi-index notation. We write

$$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \qquad \text{and} \qquad |\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n,$$

where the α_i are non-negative integers. For a vector $x = (x_1, \ldots, x_n)$ we define $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, and similarly we set

$$\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}.$$

We write $\alpha \le \beta$ if $\alpha_i \le \beta_i$ for all $i = 1, \ldots, n$, and define the factorial $\alpha! = \alpha_1! \cdots \alpha_n!$. With this notation the Leibniz formula for the differentiation of a product can be written as

$$\partial^\alpha(fg) = \sum_{0 \le \beta \le \alpha} \binom{\alpha}{\beta} (\partial^\beta f)(\partial^{\alpha-\beta} g),$$

where

$$\binom{\alpha}{\beta} = \frac{\alpha!}{\beta!(\alpha-\beta)!}$$

and in a mild abuse of notation we have written 0 for $(0, \ldots, 0)$.

Finally we write

$$|\nabla u| := \left(\sum_{i,j=1}^{3} |\partial_i u_j|^2 \right)^{1/2}.$$

1.3 Spaces of continuous and differentiable functions

Let Ω be a bounded open subset of $\mathbb{R}^n$. We use the following standard notation.

- $C^0(\Omega)$ is the space of continuous functions on Ω. The space $C^0(\overline{\Omega})$ of continuous functions on $\overline{\Omega}$ with norm

$$\|u\|_{C^0(\overline{\Omega})} := \sup_{x\in\overline{\Omega}} |u(x)|$$

 is a Banach space.
- $C^{0,\gamma}(\overline{\Omega})$, $0 < \gamma \le 1$, is the space of all uniformly γ-Hölder continuous functions on $\overline{\Omega}$, i.e. functions $f\colon \overline{\Omega} \to \mathbb{R}$ for which there exists $C > 0$ such that

$$|f(x) - f(y)| \le C|x-y|^\gamma \qquad \text{for every} \quad x, y \in \overline{\Omega}.$$

 The space $C^{0,\gamma}(\overline{\Omega})$ of γ-Hölder continuous functions on $\overline{\Omega}$ with norm

$$\|u\|_{C^0(\overline{\Omega})} + \sup_{x,y\in\overline{\Omega}} \frac{|u(x)-u(y)|}{|x-y|^\gamma}$$

 is a Banach space.

- $C^k(\Omega)$ is the space of all k-times continuously differentiable functions on Ω. The space $C^k(\overline{\Omega})$, of all k-times continuously differentiable functions with derivatives up to order k continuous on $\overline{\Omega}$ is a Banach space when equipped with the norm

$$\|u\|_{C^k(\overline{\Omega})} := \sum_{|\alpha|\le k} \|\partial^\alpha u\|_{C^0(\overline{\Omega})}.$$

- $C^{k,\gamma}(\overline{\Omega})$, $0 < \gamma \le 1$, consists of all functions $u \in C^k(\Omega)$ for which all the kth derivatives are Hölder continuous with exponent γ, i.e. $\partial^\alpha u \in C^{0,\gamma}(\overline{\Omega})$ for every multi-index α with $|\alpha| = k$.

We will also use the following spaces of smooth (infinitely differentiable) functions.

- $C_c^\infty(\Omega)$ is the space of all smooth functions with compact support in Ω,

$$C_c^\infty(\Omega) := \{\varphi : \ \varphi \in C^\infty(\Omega), \quad \operatorname{supp}\varphi \subset\subset \Omega\},$$

 where $A \subset\subset B$ is used to denote the fact that A is a compact subset of B. At times, to maintain a unified notation across all choices of domains, we will also write $C_c^\infty(\mathbb{T}^3)$, but in this case the subscript c is redundant since all functions defined on $\mathbb{T}^3$ have compact support.
- $\mathscr{S}(\mathbb{R}^n)$ is the space of Schwartz functions on $\mathbb{R}^n$, that is the space of all smooth functions such that

$$p_{k,\alpha}(u) := \sup_{x\in\mathbb{R}^n} |x|^k |\partial^\alpha u(x)|$$

is finite for every choice of $k = 0, 1, 2, \ldots$ and multi-index $\alpha \geq 0$. We denote by $\mathscr{S}'(\mathbb{R}^n)$ the space of all tempered distributions on $\mathbb{R}^n$, that is the collection of all bounded linear functionals f on $\mathscr{S}(\mathbb{R}^n)$ that are continuous in the sense that $f(u_n) \to 0$ as $n \to \infty$ if[2] $(u_n) \in \mathscr{S}(\mathbb{R}^n)$ with $p_{k,\alpha}(u_n) \to 0$ as $n \to \infty$ (for all k and α as above). For more details see Friedlander & Joshi (1999), for example.

When X is a Banach space we denote the space of all continuous functions from $[0, T]$ into X by $C([0, T]; X)$; equipped with the norm

$$\|u\|_{C([0,T];X)} := \sup_{t\in[0,T]} \|u(t)\|_X$$

this is again a Banach space. For functions in $C([0, T]; X)$ we have the following version of the Arzelà–Ascoli Theorem, which will be one of the key ingredients in the proof of a simple version of the Aubin–Lions Lemma (Theorem 4.11) that we will use to show the existence of weak solutions of the Navier–Stokes equations in Chapter 4.

Theorem 1.1 (Arzelà–Ascoli Theorem) *Let X be a Banach space and (u_n) a sequence of functions in $C([0, T]; X)$ such that*

(i) *for each $t \in [0, T]$ there exists a compact set $K(t) \subset X$ such that for every $n \in \mathbb{N}$ we have $u_n(t) \in K(t)$;*
(ii) *the functions u_n are equicontinuous: for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every $n \in \mathbb{N}$*

$$|s - t| \leq \delta \qquad \Rightarrow \qquad \|u_n(s) - u_n(t)\|_X \leq \varepsilon.$$

Then there exists a subsequence (u_{n_k}) and a function $u \in C([0, T]; X)$ such that

$$u_{n_k} \to u \quad in \quad C([0, T]; X).$$

1.4 Lebesgue spaces

Given a measurable subset Ω of $\mathbb{R}^n$, a function $f\colon \Omega \to \mathbb{R}$ is (Lebesgue) measurable if $f^{-1}(U)$ is (Lebesgue) measurable for every Borel subset U of $\mathbb{R}$. It is equivalent in this case to require that $f^{-1}(U)$ is measurable for every open set $U \subset \mathbb{R}$, or for every closed set $U \subset \mathbb{R}$.

[2] Here and in what follows we use shorthand notation $(u_n) \in X$ to denote a sequence $(u_n)_{n=1}^{\infty}$ such that $u_n \in X$ for every $n \in \mathbb{N}$.

By $L^p(\Omega)$, where $1 \le p < \infty$, we denote the standard Lebesgue space of measurable p-integrable (scalar or vector-valued) functions with the norm

$$\|u\|_{L^p} := \left(\int_\Omega |u(x)|^p \,\mathrm{d}x\right)^{1/p}.$$

The space $L^2(\Omega)$ is a Hilbert space when equipped with the inner product

$$\langle u, v\rangle := \int_\Omega u(x)\cdot v(x)\,\mathrm{d}x;$$

since we use this space so frequently we reserve the notation $\|\cdot\|$ for the L^2 norm, and usually omit the L^2 subscript. We use the same notation $\langle f, g\rangle$ to denote $\int_\Omega f(x)\cdot g(x)\,\mathrm{d}x$ whenever $f\cdot g \in L^1$ (see also Section 1.8).

The space $L^\infty(\Omega)$ of essentially bounded functions on Ω is equipped with the standard norm

$$\|u\|_{L^\infty} := \operatorname{ess\,sup}|u| = \inf\{a \in \mathbb{R} : \ \mu(\{x \in \Omega : \ |u(x)| \ge a\}) = 0\},$$

where μ denotes the Lebesgue measure.

For $1 \le p \le \infty$ the space $L^p_{\mathrm{loc}}(\Omega)$ consists of those functions that are contained in $L^p(K)$ for every compact subset K of Ω.

We now recall some elementary facts about Lebesgue spaces, and in particular mention some inequalities that will be used frequently in what follows.

Theorem 1.2 (Hölder's inequality) *Let Ω be a measurable set in $\mathbb{R}^n$, either bounded or unbounded. If $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, where*

$$\frac{1}{p} + \frac{1}{q} = 1, \qquad 1 \le p, q \le \infty,$$

then $uv \in L^1(\Omega)$ and

$$\int_\Omega |u(x)v(x)|\,\mathrm{d}x \le \|u\|_{L^p(\Omega)}\|v\|_{L^q(\Omega)}.$$

The Hölder inequality can be used in any domain since its proof is just a simple application of Young's inequality

$$ab \le \frac{a^p}{p} + \frac{b^q}{q} \qquad \text{for} \qquad \frac{1}{p} + \frac{1}{q} = 1, \quad 1 \le p, q \le \infty \tag{1.5}$$

(see Exercise 1.1 for a proof of (1.5)). We often use Young's inequality 'with ε': with (p, q) as in (1.5), for any $\varepsilon > 0$ we have

$$ab \le \varepsilon a^p + C(\varepsilon)b^q;$$

in fact $C(\varepsilon) = q^{-1}(p\varepsilon)^{-q/p}$ but this explicit form is rarely required.

Applying Hölder's inequality with $v \equiv 1$ on Ω shows that Lebesgue spaces are nested on domains of finite measure.

Corollary 1.3 *If $\Omega \subset \mathbb{R}^n$ is a set of finite measure then*

$$L^q(\Omega) \subset L^p(\Omega) \qquad \text{if} \qquad 1 \le p \le q \le \infty.$$

It is important to observe that this is not the case when Ω does not have finite measure (see Exercise 1.2).

In the context of the Navier–Stokes equations we often need to estimate integrals of products of three functions (usually the nonlinear term $(u \cdot \nabla)u$ multiplied by some 'test' function). Therefore in addition to the standard version of Hölder's inequality we will also use the following variant (for the proof see Exercise 1.3).

Theorem 1.4 *Let Ω be a measurable set in $\mathbb{R}^n$. If $u \in L^p(\Omega)$, $v \in L^q(\Omega)$, and $w \in L^r(\Omega)$ with*

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1, \qquad 1 \le p, q, r \le \infty,$$

then $uvw \in L^1(\Omega)$ and

$$\int_\Omega |u(x)v(x)w(x)| \, \mathrm{d}x \le \|u\|_{L^p} \|v\|_{L^q} \|w\|_{L^r}.$$

The two most frequent uses of this in what follows[3] will be with exponents $(6, 2, 3)$,

$$\|uvw\|_{L^1} \le \|u\|_{L^6} \|v\|_{L^2} \|w\|_{L^3}, \tag{1.6}$$

and $(4, 2, 4)$,

$$\|uvw\|_{L^1} \le \|u\|_{L^4} \|v\|_{L^2} \|w\|_{L^4}.$$

We will often estimate the L^6 norm by the norm in H^1 using the 3D Sobolev embedding $H^1(\Omega) \subset L^6(\Omega)$ (see Theorem 1.7 or Theorem 1.18), and the L^3 and L^4 norms using the very useful technique of Lebesgue interpolation from the following theorem, whose proof is another application of Hölder's inequality (see Exercise 1.4).

[3] Since we will often be estimating expressions of the form $\int |u||\nabla v||w|$ (see Exercise 1.5) we will usually want to put the L^2 norm on the gradient term; this is the reason for the ordering of the norms in the two examples here.

Theorem 1.5 (Lebesgue Interpolation) *Let Ω be a measurable set in $\mathbb{R}^n$. If $1 \le p \le r \le q \le \infty$ and $u \in L^p(\Omega) \cap L^q(\Omega)$ then $u \in L^r(\Omega)$ with*

$$\|u\|_{L^r} \le \|u\|_{L^p}^{\alpha} \|u\|_{L^q}^{1-\alpha}, \quad \text{where} \quad \frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}. \tag{1.7}$$

The case that we will use most frequently will be the interpolation of L^3 between L^2 and L^6,

$$\|u\|_{L^3} \le \|u\|_{L^2}^{1/2} \|u\|_{L^6}^{1/2}. \tag{1.8}$$

It is worth remembering that the Hölder and interpolation inequalities are valid in any measurable subset of $\mathbb{R}^n$, bounded or unbounded, and are dimension independent.

1.5 Fourier expansions

For real-valued functions defined on $\mathbb{T}^3$ it is often useful to consider their Fourier expansion, i.e. to write a function u defined on $\mathbb{T}^3$ in the form

$$u(x) = \sum_{k \in \mathbb{Z}^3} \hat{u}_k \mathrm{e}^{\mathrm{i}k\cdot x}, \tag{1.9}$$

where the coefficients $\hat{u}_k$ are complex numbers satisfying the condition

$$\hat{u}_k = \overline{\hat{u}_{-k}} \qquad \text{for all} \qquad k \in \mathbb{Z}^3, \tag{1.10}$$

imposed to ensure that u is real.

The main issue is the convergence of the series expansion in (1.9). In some cases this is relatively simple: we can write any $u \in L^2(\mathbb{T}^3)$ in the form (1.9), with the understanding that the sum converges in $L^2(\mathbb{T}^3)$, since the collection $\{\mathrm{e}^{\mathrm{i}k\cdot x}\}$ forms an orthogonal basis for (complex-valued functions in) $L^2(\mathbb{T}^3)$; if needed, the Fourier coefficients $\hat{u}_k$ can be computed explicitly via the integral

$$\hat{u}_k = \frac{1}{(2\pi)^3} \int_{\mathbb{T}^3} \mathrm{e}^{-\mathrm{i}k\cdot x} u(x)\,\mathrm{d}x, \tag{1.11}$$

using the orthogonality of the basis elements. When u is an m-component vector-valued function the coefficients $\hat{u}_k$ are elements of $\mathbb{C}^m$ and condition (1.10) holds for all components of $\hat{u}_k$.

The characterisation of functions in $L^2(\mathbb{T}^3)$ in terms of their Fourier coefficients is straightforward: $u \in L^2(\mathbb{T}^3)$ if and only if $\sum |\hat{u}_k|^2 < \infty$. However, in other L^p spaces the situation is somewhat more complicated.

Let (C_N) be an increasing sequence of subsets of $\mathbb{Z}^3$ such that

$$\bigcup_{N=1}^{\infty} C_N = \mathbb{Z}^3.$$

Then

$$\left\| u - \sum_{k \in C_N} \hat{u}_k \mathrm{e}^{\mathrm{i}k\cdot x} \right\|_{L^p} \to 0 \qquad \text{as} \qquad N \to \infty$$

for every $u \in L^p(\mathbb{T}^3)$, $1 \le p < \infty$, if and only if the maps

$$u \mapsto \sum_{k \in C_N} \hat{u}_k \mathrm{e}^{\mathrm{i}k\cdot x}$$

are uniformly bounded (in N) from $L^p(\mathbb{T}^3)$ into $L^p(\mathbb{T}^3)$. Boundedness of the 'truncated sum operators' implies convergence since trigonometric polynomials are dense in $L^p(\mathbb{T}^3)$, while the converse follows using the Principle of Uniform Boundedness (see Proposition 1.9 in Muscalu & Schlag, 2013, for example).

In the one-dimensional case (functions on $\mathbb{T}$) the Fourier expansions converge in every L^p, $1 < p < \infty$, see Theorem 3.20 and Exercise 3.7 in Muscalu & Schlag (2013), for example. In higher dimensions this can only be guaranteed when $p = 2$ (see Corollary 3.6.10 in Grafakos (2008); the proof relies on a result of Fefferman (1971) for the equivalent phenomenon in the context of the Fourier transform). However, one can obtain convergence in $L^p(\mathbb{T}^3)$, $p \neq 2$, by considering 'square sums' of Fourier components, i.e. using the one-dimensional result repeatedly.

Theorem 1.6 *Let $Q_N = [-N, N]^3 \cap \mathbb{Z}^3$. For every $u \in L^1(\mathbb{T}^3)$ and every $N \in \mathbb{N}$ define*

$$S_N(u) := \sum_{k \in Q_N} \hat{u}_k \mathrm{e}^{\mathrm{i}k\cdot x},$$

where the Fourier coefficients $\hat{u}_k$ are given by (1.11). Then for every $1 < p < \infty$ there is a constant $C_p > 0$, independent of N, such that

$$\|S_N(u)\|_{L^p} \le C_p \|u\|_{L^p} \quad \text{for all} \quad u \in L^p(\mathbb{T}^3)$$

and $S_N(u) \to u$ in $L^p(\mathbb{T}^3)$.

1.6 Sobolev spaces $W^{k,p}$

We begin with some well-known results about the Sobolev spaces $W^{k,p}$ for integer values of k. If not stated otherwise we assume below that $\Omega \subseteq \mathbb{R}^n$ is either the whole space or a bounded domain with at least Lipschitz boundary.

We recall that a function $g \in L^1_{\rm loc}(\Omega)$ is the weak derivative $\partial_i u$ of a function $u \in L^1_{\rm loc}(\Omega)$ if

$$\int_\Omega g(x)\varphi(x)\,\mathrm{d}x = -\int_\Omega u(x)\partial_i\varphi(x)\,\mathrm{d}x \qquad \text{for all} \quad \varphi \in C_c^\infty(\Omega).$$

Let $1 \le p < \infty$. The space $W^{1,p}(\Omega)$ consists of all those functions $u \in L^p(\Omega)$ all of whose first weak derivatives $\partial_i u$ exist and are in $L^p(\Omega)$ (we often write $\nabla u \in L^p(\Omega)$ for short). The norm in $W^{1,p}(\Omega)$ is given by

$$\|u\|_{W^{1,p}} := \left(\int_\Omega |u(x)|^p + \sum_{i=1}^n \int_\Omega |\partial_i u(x)|^p\,\mathrm{d}x\right)^{1/p}. \tag{1.12}$$

The space $C^\infty(\Omega) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$.

The space $W_0^{1,p}(\Omega)$, with $1 \le p < \infty$ is defined as the closure of $C_c^\infty(\Omega)$ in the norm (1.12). The space $W_0^{1,p}(\Omega)$ is the subset of $W^{1,p}(\Omega)$ that consists of functions vanishing on the boundary (in the sense of trace). Usually we have $W_0^{1,p}(\Omega) \neq W^{1,p}(\Omega)$, but on the whole space these coincide:

$$W_0^{1,p}(\mathbb{R}^n) = W^{1,p}(\mathbb{R}^n).$$

Suppose now that $k \ge 2$ and $1 \le p < \infty$. The space $W^{k,p}(\Omega)$ is the space of functions $u \in W^{k-1,p}(\Omega)$ all of whose first weak derivatives also belong to $W^{k-1,p}(\Omega)$. The standard norm in $W^{k,p}(\Omega)$ is given by

$$\|u\|_{W^{k,p}}^p := \sum_{|\alpha| \le k} \|\partial^\alpha u\|_{L^p}^p. \tag{1.13}$$

The space $W_0^{k,p}(\Omega)$ is the closure of $C_c^\infty(\Omega)$ in the $W^{k,p}$ norm.

The following theorem gives the fundamental embedding results for the spaces $W^{1,p}$, valid on any sufficiently regular domain.

Theorem 1.7 *Suppose that $u \in W^{1,p}(\Omega)$, where $\Omega = \mathbb{T}^n$ or $\Omega \subseteq \mathbb{R}^n$ (bounded or unbounded).*

(i) *Sobolev embedding: if $1 \le p < n$ then we have a continuous embedding*

$$W^{1,p}(\Omega) \subset L^{p^*}(\Omega), \qquad \text{where} \quad p^* = np/(n-p).$$