Part I

Basics

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Excerpt
More information

2 OSCILLATIONS

You don't know anything about trigonometry? That's OK, I'll teach you all you need to know as we go along. I will introduce summation notation, but explain it. If you know how to integrate, that might be an asset, but it is not strictly necessary as I will give you a visual description of what is involved. This should allow you to appreciate the meaning of the equations even if you do not have a full grasp of the apparatus of integration.

In this chapter we meet *oscillations*. We will look at the general way to describe any oscillation using mathematics.

To describe an oscillation in mathematical terms, we identify three key properties: how rapid it is, how large it is and when it starts. These three properties are more formally defined as *frequency*, *amplitude* and *initial phase*. We can express an oscillation mathematically by using a trigonometric function such as cosine or sine or by using a compact exponential notation involving complex numbers.

The *time-bandwidth theorem* appears over and over again in terahertz physics. It says that the product of the duration of a pulse (time) and the range of frequencies encompassed in the pulse (bandwidth) has a minimum value. Looked at in one way, if we have a short pulse, the pulse must involve a large range of frequencies. Looked at in another way, a well-defined frequency implies a very long pulse. This profound yet simple concept has wide-ranging ramifications in the production, detection and application of terahertz-frequency electromagnetic radiation.

Fourier methods play a huge role in modern physics and engineering. Here is the basic concept in a nutshell: an arbitrary oscillation can be constructed from fundamental oscillations. The Fourier method can be applied coming or going. In *Fourier synthesis*, one or more fundamental oscillations are added together to produce the final desired oscillation (which can be more or less anything you want). In *Fourier analysis*, a given, and possibly quite complicated, oscillation is broken up into its constituent fundamental oscillations. Both Fourier synthesis and Fourier analysis find many applications in terahertz physics. A good appreciation of their power is essential; an appreciation of their beauty is optional.

Learning goals

There are three important things I want you to know by the time you finish this chapter:

- how to describe oscillations in mathematical terms,

- the meaning of the time-bandwidth theorem,
- the importance of Fourier methods.

2.1 Describing oscillations

Let us start with a very simple example. About the simplest oscillation I can think of is something turning on and off. Something like a flashing light. The sequence on, off, on, off, on, off, on, repeats over and over.

Figure 2.1 represents graphically the very simple oscillation of a light switching on and off.

Let us observe first that the phenomenon I have chosen to describe is **periodic**: the same thing keeps happening over and over again. The prime characteristic of this phenomenon is its **frequency** (Figure 2.1a). The frequency is the number of oscillations per unit time. In this example the frequency happens to be 5 oscillations per minute. This frequency corresponds to the light being on for 6 seconds, then off for 6 seconds, and so on (Figure 2.1b).

Let us now elaborate on this example a little. Let us now consider measuring the amount of light more accurately, rather than just saying the light is 'on' or 'off'. Let's say we found a meter and measured the light to produce an illuminance of 100 lux. Then we could represent the oscillation more fully as in Figure 2.1c and Figure 2.1d, where a label is added to each vertical axis to denote the size of the oscillation. The size of the oscillation is its **amplitude**. More precisely, the amplitude is measured as the swing above and below the average value of the oscillation. In this example, the average value is 50 lux, and the illuminance swings 50 lux above and below this. So the amplitude is formally defined as 50 lux in this example (Figure 2.1d), not 100 lux. The amplitude has units that depend on the quantity being measured. For example, the temperature in a room might be oscillating and the amplitude would then be measured in temperature units, such as Celsius degrees. In the case of terahertz phenomena we are often interested in the amplitude of an electric field, and this is measured in the units of volts per metre.

Now look at Figure 2.1e and Figure 2.1f, where a different oscillation is shown. This second oscillation corresponds to a second light being turned off and on repeatedly. The oscillation in Figure 2.1e/f differs from the oscillation in Figure 2.1c/d in three key respects. Pay attention, for these are the three fundamental characteristics of an oscillation. First, the two oscillations differ in *frequency*. We have seen already that the first oscillation has a rate or frequency of 5 per minute. The second oscillation has a rate of 3 per minute (Figure 2.1e). This smaller rate corresponds to a greater time for the second lamp being turned on and off; it is off for 10 seconds, on for 10 seconds, and so on (Figure 2.1f). Second, the two oscillations differ in size, in *amplitude*. We have seen already that the first oscillation has an amplitude of 50 lux. The second oscillation has an amplitude of 150 lux. Third, the two oscillations *start at different points*. Taking the far left of Figure 2.1d and of Figure 2.1f to be the beginning of our measurement,

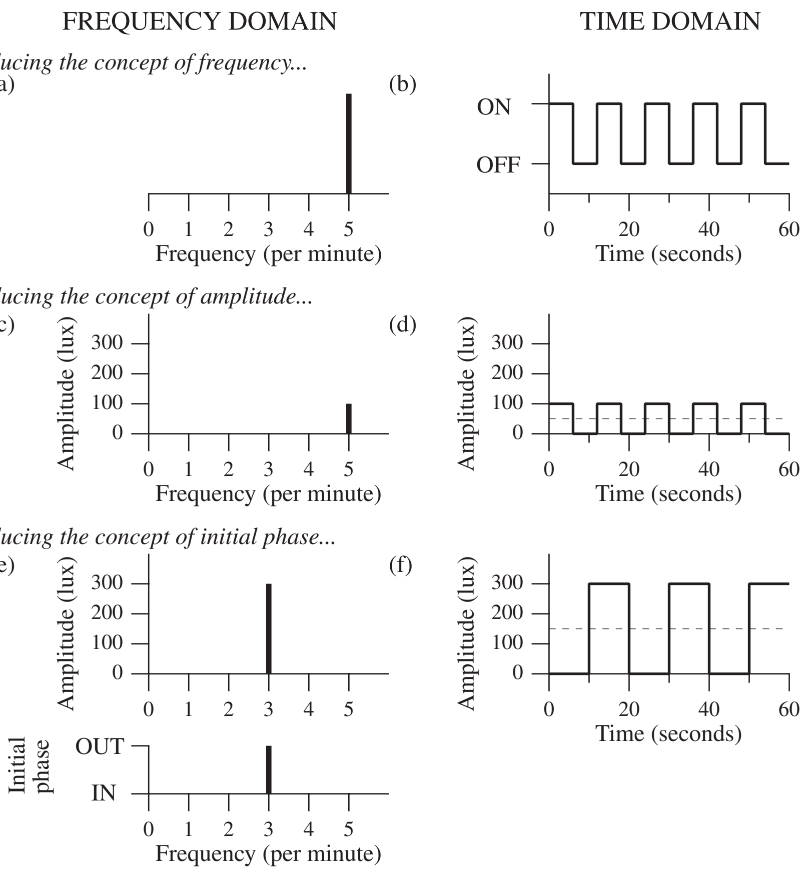

Figure 2.1 Describing oscillations. The figure introduces the three key properties of an oscillation: *frequency*, *amplitude* and *initial phase*. These properties are illustrated in both the frequency domain (left-hand side of the figure) and in the time domain (right-hand side of the figure). Panel (a) represents an occurrence with a *frequency* of 5 times per minute, such as (b) a light being repeatedly turned on for 6 seconds then off for 6 seconds. Panels (c) and (d) now specify the *amplitude* of the phenomena – the light provides an illuminance of 100 lux. (A lamp in your house might give this much illumination.) Panels (e) and (f) represent another light, giving an illuminance of 300 lux, first being turned off for 10 seconds, then on for 10 seconds, and so on. This oscillation differs from the previous oscillation in all three key characteristics of frequency, amplitude, and *initial phase*.

we see the first oscillation starts at the value of 100 lux, whereas the second starts at the value of zero. Had we only one oscillation, we may not have noticed this subtlety, but with two oscillations, we see they are out of step at the outset. The **initial phase** tells us where we are along the complete cycle of an oscillation at a reference time. The reference time usually chosen is the time when the measurement begins. Seeing the two oscillations helps us recognise there is a difference in initial phase, but we do not need two oscillations; we can define initial phase for a single oscillation by noting how far along a complete cycle we are at the reference time. I have been assuming that the first oscillation has zero phase, meaning that I count the oscillation as beginning when the light is switched on. I have not explicitly shown this zero initial phase in Figure 2.1c. The second light is halfway through its cycle when we start the measurement (Figure 2.1f); we have to wait 10 seconds, or half a cycle, before it is switched on. I count this as being completely out of phase. The phase of the second oscillation is indicated on a second vertical axis in Figure 2.1e.

We can choose to describe oscillations in the **frequency domain** (the left-hand side of Figure 2.1). In the frequency domain, we see explicitly what different frequencies are present, their relative amplitudes and initial phases. Alternately, we may describe oscillations in the **time domain** (the right-hand side of Figure 2.1). Oscillations shown in the frequency domain and the time domain amount to the same thing; equivalent information is in each representation. However, depending on our purpose, working in one or other domain may prove to be more convenient. This book is about phenomena characterised by particular (terahertz) frequencies, so prepare to spend some time in the (terahertz) frequency domain.

Let us now turn to a smoother example. A common form of periodic motion is rotation. We will start with the simplest version of rotation – motion on the simplest curve (the circle) and at the simplest rate (evenly). Such motion is called **uniform circular motion**. I expect you have studied uniform circular motion previously but it doesn't matter if you haven't. To a reasonable approximation, uniform circular motion may be used to describe the motion of the earth around the sun, or the motion of an electron around a proton in a hydrogen atom. Uniform circular motion is easy to grasp. In Figure 2.2a I show uniform circular motion by representing a particle moving in – you guessed it – a circle, and at – wait for it – a steady speed.

The situation is very simple, as described first in words (uniform circular motion) then described in a picture (Figure 2.2a). We may also describe this situation very simply using a third language, mathematics. Let's call the angle through which the particle has moved around the circle theta, θ. The first symbol I think of when I think of an angle is θ, so that is what I use here. Throughout the book, θ is employed as a general-purpose angle. We will measure the angle in radians. If you have not previously met the radian, abbreviated *rad*, all you need to know is it is used to measure angles around a circle, and moving once around the circle amounts to 2π radians (Figure 2.2b). More details on angles and their measurement can be found in Appendix C. Of course, as the particle is moving, the angle is always changing. But as the particle is moving uniformly, the angle is changing at a steady rate. Let's call this steady rate of angular change ω. Assuming we start with θ being zero at time zero then at time t

$$\theta = \omega t. \tag{2.1}$$

This relationship is illustrated in Figure 2.2c. For example, if the rate of change of angle θ is one twelfth of a circle per second, or $2\pi/12 = \pi/6$ radians per second, after 1, 2 and

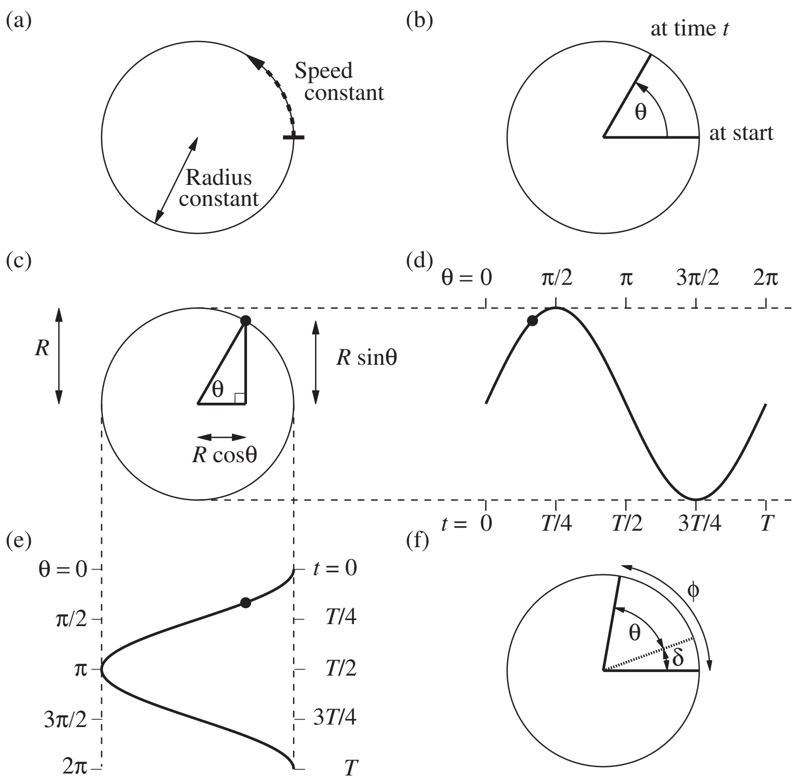

Figure 2.2 Uniform circular motion. (a) Motion at a steady rate around a circular path, in other words, uniform circular motion. (b) In a time interval t, motion at the steady angular rate of ω results in an angle of $\theta = \omega t$ being traversed. Here t is measured in seconds, ω is in radians per second and θ is in radians. The time taken to complete one lap of the circle is the period, T, and is measured in seconds. The period is directly related to the number of laps per second, or frequency, f, by $f = 1/T$. Frequency has units of hertz. The idea of laps per second may be expressed in terms of radians per second; there are 2π radians in one lap. The angular frequency is then $\omega = 2\pi/T = 2\pi f$. (c) Defining the trigonometric functions cosine and sine. (d, e) Uniform circular motion projected onto two perpendicular axes. (f) If the motion does not start at the usual origin, this is accommodated by an initial phase δ. The phase, ϕ, takes into account both the motion described previously (θ) and the initial phase (δ); $\phi = \theta + \delta$.

3 seconds, the particle will have moved one twelfth, then two twelfths (one sixth) and finally three twelfths (one quarter) of the way around the circle, Figure 2.2c.

Once the angle changes by a full circle, or $\theta = 2\pi$, the motion starts again. Putting $\theta = 2\pi$ into Equation (2.1), the time taken for one complete oscillation is $2\pi/\omega$. Since the frequency is the inverse of the time taken for one complete oscillation, $f = \omega/2\pi$. Making ω the subject of the formula,

$$\omega = 2\pi f. \tag{2.2}$$

We call ω the **angular frequency**. (Earlier I called this the steady rate of angular change; it amounts to the same thing.) Rather than express θ in terms of angular frequency we can express it in terms of frequency by substituting Equation (2.2) into Equation (2.1):

$$\theta = 2\pi f t. \tag{2.3}$$

By definition, being always on the circle, the distance from the centre of the circle is always the same. We will call this distance R.

$$R = \text{constant}. \tag{2.4}$$

The two-dimensional (or *plane*) motion about a centre (or *pole*) is conveniently expressed in the **plane-polar coordinates**, R and θ, as introduced in Equations (2.4) and (2.1).

In many circumstances, we like to have coordinates straight and true: not the mongrel distance and angle of the plane-polar system. We want to work in Cartesian **rectangular coordinates**, x and y. So our task is now to express the position of the particle at any time in x and y coordinates. Let us break up this job into two parts, first the x coordinate, then the y. To begin with (in other words, when time $t = 0$), the particle is actually on the x axis, and a distance R from the centre of the circle (the origin of the x-y axis system). So the x component is R. After the particle has moved through $\pi/3$ radians, which happens after 2 seconds in our example, the x component is $R/2$. (You could take my word for it, or measure it.) We could do the same for every angle – write down the projection onto the x axis (Figure 2.2e). Doing this produces the cosine function. In mathematical shorthand,

$$x = R\cos\theta. \tag{2.5}$$

If you have never met the cosine function before, note that this equation defines it. In words, for a given angle of rotation anticlockwise around a circle starting at the x axis, the cosine function gives the projection of the position onto the x axis (as a proportion of the radius). In a similar way, we can write the y component as

$$y = R\sin\theta. \tag{2.6}$$

In words, the sine function gives the y projection of the particle position as a proportion of the radius (Figure 2.2d). More detail about the cosine and sine functions is given in Appendix C.

If we want to show the time dependence explicitly, we may use Equation (2.1) to write

$$x = R\cos\omega t \quad \text{and} \quad y = R\sin\omega t, \tag{2.7}$$

or Equation (2.3) to write

$$x = R\cos 2\pi f t \quad \text{and} \quad y = R\sin 2\pi f t. \tag{2.8}$$

If we are only interested in the projection along a particular axis, we can use one (or other) of these two equations. Some prefer the cosine expression, and refer to cosinusoidal oscillations. Some prefer the sine expression, and refer to sinusoidal oscillations.

It really amounts to the same thing, as the cosine function and the sine function differ only by a rotation of one-quarter of a circle, in other words, by $\pi/2$ radians (Appendix C):

$$\cos\theta = \sin(\theta + \pi/2). \tag{2.9}$$

Thus, we may say cosine and sine differ only in initial phase. This is illustrated in Figure 2.2d/e. If we count the oscillation as beginning with the value 1 and decreasing from there, the cosine function is in phase and the sine function is one-quarter cycle, or $\pi/2$ radians, out of phase.

The discussion about the similarity between cosine and sine, apart from the starting point, brings us to the relatively minor matter of dealing with motion that does not start on the x axis at $t = 0$. We may regard this as either an offset in time or an offset in angle. (An example is the alternate starting point in Figure 2.2f.) The offset angle, δ, is how we will usually express initial phase from now on. This offset in angle corresponds to an offset in time, which we will denote by t_δ. The relation between the two quantities is

$$\delta = \omega t_\delta. \tag{2.10}$$

(This may be seen to follow from Equation (2.1).) The x value of an oscillation of initial phase δ (such as the oscillation shown in Figure 2.2f measured from the alternate starting point) may be represented by

$$x = R\cos(\omega t + \delta) = R\cos(2\pi f t + \delta). \tag{2.11}$$

The argument of a trigonometric function is called the **phase** and denoted ϕ. Note that the phase, ϕ, is distinct from the initial phase, δ. Here the phase is the argument of the cosine function,

$$\phi = \omega t + \delta = 2\pi f t + \delta. \tag{2.12}$$

Using this expression for phase we can write Equation (2.11) as

$$x = R\cos\phi, \tag{2.13}$$

putting it into exactly the same form as Equation (2.5).

So far, the physical quantity we have been focussing on has been the position in space. We can extend the idea of oscillations to other quantities, such as temperature, or pressure, or electric field. Using A to represent a general quantity and A_0 the amplitude of that quantity, we can write a **harmonic** oscillation in general as

$$A = A_0\cos(\omega t + \delta) = A_0\cos(2\pi f t + \delta). \tag{2.14}$$

Note that A and A_0 have the same dimensions so that the equation balances.

To sum up, the three main characteristics of an oscillation are its *frequency*, its *amplitude* and its *initial phase*. In mathematical terms, beginning from the idea of motion in a circle, we may write these three key parameters for circular, or harmonic oscillations as

Frequency	f	hertz	(2.15)
Amplitude	A_0	(units pertinent to the oscillation)	(2.16)
Initial phase	δ	radians	(2.17)

A general mathematical form to represent any harmonic oscillation is then:

$$A = A_0 \cos(2\pi f t + \delta) = A_0 \cos \phi. \tag{2.18}$$

The constants are A_0, f and δ, the three characteristics of the oscillation; the variable is the time, t; these four quantities are related to the size of the oscillation via the trigonometric function cosine. The phase, ϕ, incorporates the change with time, $2\pi f t$, and the initial phase, δ.

Example 2.1 The unit diamond

(This example assumes a knowledge of trigonometry. You can skip it if your trigonometric knowledge is weak or non-existent.) The definitions of cosine and sine were introduced in Figure 2.2 based on motion in a circular path. In a similar fashion, other functions may be defined based on motion along other paths. Consider motion along a *unit diamond* (Figure 2.3a); that is, a set of four straight lines running from (1, 0) to (0, 1), to (−1, 0), to (0, −1) and back to (1, 0). Give an expression for the projection of a position on the unit diamond onto the y axis as a function of angle θ (measured anticlockwise from the x axis). Also give an expression for the projection onto the x axis.

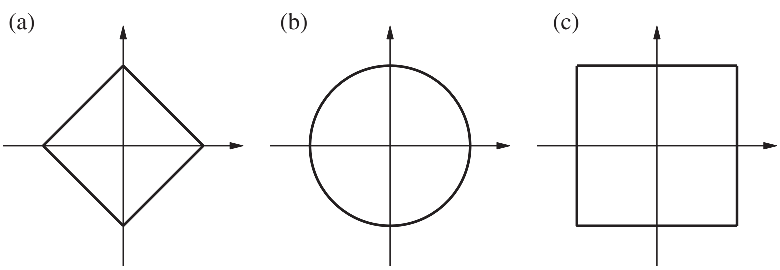

Figure 2.3 (a) The unit diamond. (b) The unit circle. (c) The unit square. The figures cross the axes at a unit distance from the origin.