CAMBRIDGE TRACTS IN MATHEMATICS

General Editors

B. BOLLOBÁS, W. FULTON, A. KATOK, F. KIRWAN,
P. SARNAK, B. SIMON, B. TOTARO

192 Normal Approximations with Malliavin Calculus
CAMBRIDGE TRACTS IN MATHEMATICS

GENERAL EDITORS
B. BOLLOBÁS, W. FULTON, A. KATOK, F. KIRWAN, P. SARNAK,
B. SIMON, B. TOTARO

A complete list of books in the series can be found at www.cambridge.org/mathematics. Recent titles include the following:

157. Affine Hecke Algebras and Orthogonal Polynomials. By I. G. MACDONALD
158. Quasi-Frobenius Rings. By W. K. NICHOLSON and M. F. YOUSIF
159. The Geometry of Total Curvature on Complete Open Surfaces. By K. SHIOHAMA,
T. SHIOYA, and M. TANAKA
160. Approximation by Algebraic Numbers. By Y. BUGEAUD
162. Lévy Processes in Lie Groups. By M. LIAO
163. Linear and Projective Representations of Symmetric Groups. By A. KLESHCHEV
164. The Covering Property Axiom, CPA. By K. CIESIELSKI and J. PAWLIKOWSKI
165. Projective Differential Geometry Old and New. By V. OVSIEHNKO and S. TABACHNIKOV
166. The Lévy Laplacian. By M. N. FELLER
D. MEYER and L. SMITH
168. The Cube-A Window to Convex and Discrete Geometry. By C. ZONG
169. Quantum Stochastic Processes and Noncommutative Geometry. By K. B. SINHA and
D. GOSWAMI
170. Polynomials and Vanishing Cycles. By M. TIBAR
171. Orbifolds and Stringy Topology. By A. ADEM, J. LEIDA, and Y. RUAN
172. Rigid Cohomology. By B. LE STUM
173. Enumeration of Finite Groups. By S. R. BLACKBURN, P. M. NEUMANN, and
G. VENKATARAMAN
174. Forcing Idealized. By J. ZAPLETAL
175. The Large Sieve and its Applications. By E. KOWALSKI
176. The Monster Group and Majorana Involutions. By A. A. IVANOV
177. A Higher-Dimensional Sieve Method. By H. G. DIAMOND, H. HALBERSTAM, and
W. F. GALWAY
178. Analysis in Positive Characteristic. By A. N. KOCHUBEI
179. Dynamics of Linear Operators. By F. BAYART and É. MATHERON
180. Synthetic Geometry of Manifolds. By A. KOCK
181. Totally Positive Matrices. By A. PINKUS
184. Algebraic Theories. By J. ADÁMEK, J. ROSSICKY, and E. M. VITALE
A. KATOK and V. NÎTICĂ
186. Dimensions, Embeddings, and Attractors. By J. C. ROBINSON
188. Modern Approaches to the Invariant Subspace Problem. By I. CHALENDAR and
J. R. PARTINGTON
191. Malliavin Calculus for Lévy Processes and Infinite-Dimensional Brownian Motion. By
H. OSSWALD
Normal Approximations with Malliavin Calculus
From Stein’s Method to Universality

IVAN NOURDIN
Université de Nancy I, France

GIOVANNI PECCATI
Université du Luxembourg
To Lili, Juliette and Delphine.
To Emma Eliza and Ieva.
Contents

Preface xi

Introduction 1

1 Malliavin operators in the one-dimensional case 4
   1.1 Derivative operators 4
   1.2 Divergences 8
   1.3 Ornstein–Uhlenbeck operators 9
   1.4 First application: Hermite polynomials 13
   1.5 Second application: variance expansions 15
   1.6 Third application: second-order Poincaré inequalities 16
   1.7 Exercises 19
   1.8 Bibliographic comments 20

2 Malliavin operators and isonormal Gaussian processes 22
   2.1 Isonormal Gaussian processes 22
   2.2 Wiener chaos 26
   2.3 The derivative operator 28
   2.4 The Malliavin derivatives in Hilbert spaces 32
   2.5 The divergence operator 33
   2.6 Some Hilbert space valued divergences 35
   2.7 Multiple integrals 36
   2.8 The Ornstein–Uhlenbeck semigroup 45
   2.9 An integration by parts formula 53
   2.10 Absolute continuity of the laws of multiple integrals 54
   2.11 Exercises 55
   2.12 Bibliographic comments 57

3 Stein’s method for one-dimensional normal approximations 59
   3.1 Gaussian moments and Stein’s lemma 59
   3.2 Stein’s equations 62
## Contents

3.3 Stein’s bounds for the total variation distance 63  
3.4 Stein’s bounds for the Kolmogorov distance 65  
3.5 Stein’s bounds for the Wasserstein distance 67  
3.6 A simple example 69  
3.7 The Berry–Esseen theorem 70  
3.8 Exercises 75  
3.9 Bibliographic comments 78

### 4 Multidimensional Stein’s method

4.1 Multidimensional Stein’s lemmas 79  
4.2 Stein’s equations for identity matrices 81  
4.3 Stein’s equations for general positive definite matrices 84  
4.4 Bounds on the Wasserstein distance 85  
4.5 Exercises 86  
4.6 Bibliographic comments 88

### 5 Stein meets Malliavin: univariate normal approximations

5.1 Bounds for general functionals 89  
5.2 Normal approximations on Wiener chaos 93  
5.3 Normal approximations in the general case 102  
5.4 Exercises 108  
5.5 Bibliographic comments 115

### 6 Multivariate normal approximations

6.1 Bounds for general vectors 116  
6.2 The case of Wiener chaos 120  
6.3 CLTs via chaos decompositions 124  
6.4 Exercises 126  
6.5 Bibliographic comments 127

### 7 Exploring the Breuer–Major theorem

7.1 Motivation 128  
7.2 A general statement 129  
7.3 Quadratic case 133  
7.4 The increments of a fractional Brownian motion 138  
7.5 Exercises 145  
7.6 Bibliographic comments 146

### 8 Computation of cumulants

8.1 Decomposing multi-indices 148  
8.2 General formulae 149  
8.3 Application to multiple integrals 154
Contents

8.4 Formulae in dimension one 157
8.5 Exercises 159
8.6 Bibliographic comments 159

9 Exact asymptotics and optimal rates 160
  9.1 Some technical computations 160
  9.2 A general result 161
  9.3 Connections with Edgeworth expansions 163
  9.4 Double integrals 165
  9.5 Further examples 166
  9.6 Exercises 168
  9.7 Bibliographic comments 169

10 Density estimates 170
  10.1 General results 170
  10.2 Explicit computations 174
  10.3 An example 175
  10.4 Exercises 176
  10.5 Bibliographic comments 178

11 Homogeneous sums and universality 179
  11.1 The Lindeberg method 179
  11.2 Homogeneous sums and influence functions 182
  11.3 The universality result 185
  11.4 Some technical estimates 188
  11.5 Proof of Theorem 11.3.1 194
  11.6 Exercises 195
  11.7 Bibliographic comments 196

Appendix A Gaussian elements, cumulants and Edgeworth expansions 197
  A.1 Gaussian random variables 197
  A.2 Cumulants 198
  A.3 The method of moments and cumulants 202
  A.4 Edgeworth expansions in dimension one 203
  A.5 Bibliographic comments 204

Appendix B Hilbert space notation 205
  B.1 General notation 205
  B.2 $L^2$ spaces 205
  B.3 More on symmetrization 205
  B.4 Contractions 206
# Contents

<table>
<thead>
<tr>
<th>Appendix</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>C.5</td>
<td>Random elements</td>
<td>208</td>
</tr>
<tr>
<td>B.6</td>
<td>Bibliographic comments</td>
<td>208</td>
</tr>
<tr>
<td>C.1</td>
<td>General definitions</td>
<td>209</td>
</tr>
<tr>
<td>C.2</td>
<td>Some special distances</td>
<td>210</td>
</tr>
<tr>
<td>C.3</td>
<td>Some further results</td>
<td>211</td>
</tr>
<tr>
<td>C.4</td>
<td>Bibliographic comments</td>
<td>214</td>
</tr>
<tr>
<td>D.1</td>
<td>Definition and immediate properties</td>
<td>215</td>
</tr>
<tr>
<td>D.2</td>
<td>Hurst phenomenon and invariance principle</td>
<td>218</td>
</tr>
<tr>
<td>D.3</td>
<td>Fractional Brownian motion is not a semimartingale</td>
<td>221</td>
</tr>
<tr>
<td>D.4</td>
<td>Bibliographic comments</td>
<td>224</td>
</tr>
<tr>
<td>E.1</td>
<td>Dense subsets of an $L^q$ space</td>
<td>225</td>
</tr>
<tr>
<td>E.2</td>
<td>Rademacher’s theorem</td>
<td>226</td>
</tr>
<tr>
<td>E.3</td>
<td>Bibliographic comments</td>
<td>226</td>
</tr>
</tbody>
</table>

References  
Author index  
Notation index  
Subject index
Preface

This is a text about probabilistic approximations, which are mathematical statements providing estimates of the distance between the laws of two random objects. As the title suggests, we will be mainly interested in approximations involving one or more normal (equivalently called Gaussian) random elements. Normal approximations are naturally connected with central limit theorems (CLTs), i.e. convergence results displaying a Gaussian limit, and are one of the leading themes of the whole theory of probability.

The main thread of our text concerns the normal approximations, as well as the corresponding CLTs, associated with random variables that are functionals of a given Gaussian field, such as a (fractional) Brownian motion on the real line. In particular, a pivotal role will be played by the elements of the so-called Gaussian Wiener chaos. The concept of Wiener chaos generalizes to an infinite-dimensional setting the properties of the Hermite polynomials (which are the orthogonal polynomials associated with the one-dimensional Gaussian distribution), and is now a crucial object in several branches of theoretical and applied Gaussian analysis.

The cornerstone of our book is the combination of two probabilistic techniques, namely the Malliavin calculus of variations and Stein’s method for probabilistic approximations.

The Malliavin calculus of variations is an infinite-dimensional differential calculus, whose operators act on functionals of general Gaussian processes. Initiated by Paul Malliavin (starting from the seminal paper [69], which focused on a probabilistic proof of Hörmander’s ‘sum of squares’ theorem), this theory is based on a powerful use of infinite-dimensional integration by parts formulae. Although originally exploited for studying the regularity of the laws of Wiener functionals (such as the solutions of stochastic differential equations), the scope of its actual applications, ranging from density estimates to concentration inequalities, and from anticipative stochastic calculus to the
computations of ‘Greeks’ in mathematical finance, continues to grow. For a classic presentation of this subject, the reader can consult the three texts by Malliavin [70], Nualart [98] and Janson [57]. Our book is the first monograph providing a self-contained introduction to Malliavin calculus from the specific standpoint of limit theorems and probabilistic approximations.

Stein’s method can be roughly described as a collection of probabilistic techniques for assessing the distance between probability distributions by means of differential operators. This approach was originally developed by Charles Stein in the landmark paper [135], and then further refined in the monograph [136]. In recent years, Stein’s method has become one of the most popular and powerful tools for computing explicit bounds in probabilistic limit theorems, with applications to fields as diverse as random matrices, random graphs, probability on groups and spin glasses (to name but a few). The treatise [22], by Chen, Goldstein and Shao, provides an exhaustive discussion of the theoretical foundations of Stein’s method for normal approximations, as well as an overview of its many ramifications and applications (see also the two surveys by Chen and Shao [23] and Reinert [117]).

We shall show that the integration by parts formulae of Malliavin calculus can be fruitfully combined with the differential operators arising in Stein’s method. This interaction will be exploited to produce a set of flexible and far-reaching tools, allowing general CLTs (as well as explicit rates of convergence) to be deduced for sequences of functionals of Gaussian fields. It should be noted that the theory developed in this book virtually replaces every technique previously used to establish CLTs for Gaussian-subordinated random variables, e.g. those based on moment/cumulant computations (see, for example, Peccati and Taqqu [110]).

As discussed at length in the text, the theoretical backbone of the present monograph originates from the content of five papers.

– Nualart and Peccati [101] give an exhaustive (and striking) characterization of CLTs inside a fixed Wiener chaos. This result, which we will later denote as the ‘fourth-moment theorem’, yields a drastic simplification of the classic method of moments and cumulants, and is one of the main topics discussed in the book.

– Peccati and Tudor [111] provide multidimensional extensions of the findings of [101]. In view of the Wiener–Itô chaotic representation property (see Chapter 2), the findings of [111] pave the way for CLTs involving general functionals of Gaussian fields (not necessarily living inside a fixed Wiener chaos).

– The paper by Nualart and Ortiz-Latorre [100] contains a crucial methodological breakthrough, linking CLTs on Wiener chaos to the
asymptotic behavior of Malliavin operators. In particular, a prominent role is played by the norms of the Malliavin derivatives of multiple Wiener–Itô integrals.

- Nourdin and Peccati [88] establish the above-mentioned connection between Malliavin calculus and Stein’s method, thus providing substantial refinements of the findings of [100, 101, 111]. Along similar lines, the multivariate case is dealt with by Nourdin, Peccati and Réveillac [95].
- Nourdin, Peccati and Reinert [94] link the above results to the so-called universality phenomenon, according to which the asymptotic behavior of (correctly rescaled) large random systems does not depend on the distribution of their components. Universality results, also known as ’invariance principles’, are almost ubiquitous in probability: distinguished examples are the classic central limit theorem and the circular and semicircular laws in random matrix theory. See Chapter 11 for further discussions.

The above-mentioned references have been the starting point of many developments and applications. These include density estimates, concentration inequalities, Berry–Esseen bounds for power variations of Gaussian-subordinated processes, normalization of Brownian local times, random polymers, random matrices, parametric estimation in fractional models and the study of polyspectra associated with stationary fields on homogeneous spaces. Several of these extensions and applications are explicitly described in our book. See the webpage

http://www.iecn.u-nancy.fr/~nourdin/steinmalliavin.htm

for a constantly updated reference list. See the monographs [74] and [110], respectively, for further applications to random fields on the sphere (motivated by cosmological data analysis), and for a discussion of the combinatorial structures associated with the Gaussian Wiener chaos.

The book is addressed to researchers and graduate students in probability and mathematical statistics, wishing to acquire a thorough knowledge of modern Gaussian analysis as used to develop asymptotic techniques related to normal approximations.

With very few exceptions (where precise references are given), every result stated in the book is proved (sometimes through a detailed exercise), and even the most basic elements of Malliavin calculus and Stein’s method are motivated, defined and studied from scratch. Several proofs are new, and each chapter contains a set of exercises (with some hints!), as well as a number of bibliographic comments. Due to these features, the text is more or less self-contained, although our ideal reader should have attended a basic course...
Preface

in modern probability (corresponding, for example, to the books by Billingsley [13] and Chung [25]), and also have some knowledge of functional analysis (covering, for example, the content of chapters 5–6 in Dudley’s book [32]). Some facts and definitions concerning operator theory are used – we find that a very readable reference in this respect is [47], by Hirsch and Lacombe.

Acknowledgements. We heartily thank Simon Campese, David Nualart and Mark Podolskij for a careful reading of some earlier drafts of the book. All remaining errors are, of course, their sole responsibility.

Ivan Nourdin, Nancy
Giovanni Peccati, Luxembourg