

Cambridge University Press

978-1-107-01614-9 - Malliavin Calculus for Lévy Processes and Infinite-Dimensional Brownian Motion: An Introduction

Horst Osswald

Frontmatter

[More information](#)

CAMBRIDGE TRACTS IN MATHEMATICS

General Editors

B. BOLLOBÁS, W. FULTON, A. KATOK,
F. KIRWAN, P. SARNAK, B. SIMON, B. TOTARO

**191 Malliavin Calculus for Lévy Processes and
Infinite-Dimensional Brownian Motion**

Cambridge University Press

978-1-107-01614-9 - Malliavin Calculus for Lévy Processes and Infinite-Dimensional Brownian Motion: An Introduction

Horst Osswald

Frontmatter

[More information](#)

CAMBRIDGE TRACTS IN MATHEMATICS

GENERAL EDITORS

B. BOLLOBÁS, W. FULTON, A. KATOK, F. KIRWAN, P. SARNAK,
B. SIMON, B. TOTARO

A complete list of books in the series can be found at www.cambridge.org/mathematics.
Recent titles include the following:

154. Finite Packing and Covering. By K. BÖRÖCZKY, JR
155. The Direct Method in Soliton Theory. By R. HIROTA. Edited and translated by A. NAGAI, J. NIMMO, and C. GILSON
156. Harmonic Mappings in the Plane. By P. DUREN
157. Affine Hecke Algebras and Orthogonal Polynomials. By I. G. MACDONALD
158. Quasi-Frobenius Rings. By W. K. NICHOLSON and M. F. YOUSIF
159. The Geometry of Total Curvature on Complete Open Surfaces. By K. SHIOHAMA, T. SHIOYA, and M. TANAKA
160. Approximation by Algebraic Numbers. By Y. BUGEAUD
161. Equivalence and Duality for Module Categories. By R. R. COLBY and K. R. FULLER
162. Lévy Processes in Lie Groups. By M. LIAO
163. Linear and Projective Representations of Symmetric Groups. By A. KLESHCHEV
164. The Covering Property Axiom, CPA. By K. CIESIELSKI and J. PAWLIKOWSKI
165. Projective Differential Geometry Old and New. By V. OVSIENKO and S. TABACHNIKOV
166. The Lévy Laplacian. By M. N. FELLER
167. Poincaré Duality Algebras, Macaulay's Dual Systems, and Steenrod Operations. By D. MEYER and L. SMITH
168. The Cube-A Window to Convex and Discrete Geometry. By C. ZONG
169. Quantum Stochastic Processes and Noncommutative Geometry. By K. B. SINHA and D. GOSWAMI
170. Polynomials and Vanishing Cycles. By M. TIBĀR
171. Orbifolds and Stringy Topology. By A. ADEM, J. LEIDA, and Y. RUAN
172. Rigid Cohomology. By B. LE STUM
173. Enumeration of Finite Groups. By S. R. BLACKBURN, P. M. NEUMANN, and G. VENKATARAMAN
174. Forcing Idealized. By J. ZAPLETAL
175. The Large Sieve and its Applications. By E. KOWALSKI
176. The Monster Group and Majorana Involutions. By A. A. IVANOV
177. A Higher-Dimensional Sieve Method. By H. G. DIAMOND, H. HALBERSTAM, and W. F. GALWAY
178. Analysis in Positive Characteristic. By A. N. KOCHUBEI
179. Dynamics of Linear Operators. By F. BAYART and É. MATHERON
180. Synthetic Geometry of Manifolds. By A. KOCK
181. Totally Positive Matrices. By A. PINKUS
182. Nonlinear Markov Processes and Kinetic Equations. By V. N. KOLOKOLTSOV
183. Period Domains over Finite and p -adic Fields. By J.-F. DAT, S. ORLIK, and M. RAPOPORT
184. Algebraic Theories. By J. ADÁMEK, J. ROSICKÝ, and E. M. VITALE
185. Rigidity in Higher Rank Abelian Group Actions I: Introduction and Cocycle Problem. By A. KATOK and V. NIȚIȚĂ
186. Dimensions, Embeddings, and Attractors. By J. C. ROBINSON
187. Convexity: An Analytic Viewpoint. By B. SIMON
188. Modern Approaches to the Invariant Subspace Problem. By I. CHALENDAR and J. R. PARTINGTON
189. Nonlinear Perron–Frobenius Theory. By B. LEMMENS and R. NUSSBAUM
190. Jordan Structures in Geometry and Analysis. By C.-H. CHU
191. Malliavin Calculus for Lévy Processes and Infinite-Dimensional Brownian Motion. By H. OSSWALD
192. Normal Approximations with Malliavin Calculus. By I. NOURDIN and G. PECCATI

Cambridge University Press

978-1-107-01614-9 - Malliavin Calculus for Lévy Processes and Infinite-Dimensional Brownian Motion: An Introduction

Horst Osswald

Frontmatter

[More information](#)

Malliavin Calculus for Lévy Processes and Infinite-Dimensional Brownian Motion

An Introduction

HORST OSSWALD

Universität München



CAMBRIDGE
UNIVERSITY PRESS

Cambridge University Press

978-1-107-01614-9 - Malliavin Calculus for Lévy Processes and Infinite-Dimensional Brownian Motion: An Introduction

Horst Osswald

Frontmatter

[More information](#)

CAMBRIDGE UNIVERSITY PRESS

Cambridge, New York, Melbourne, Madrid, Cape Town,
Singapore, São Paulo, Delhi, Mexico City

Cambridge University Press

The Edinburgh Building, Cambridge CB2 8RU, UK

Published in the United States of America by Cambridge University Press, New York

www.cambridge.org

Information on this title: www.cambridge.org/9781107016149

© Horst Osswald 2012

This publication is in copyright. Subject to statutory exception and to the provisions of relevant collective licensing agreements, no reproduction of any part may take place without the written permission of Cambridge University Press.

First published 2012

Printed in the United Kingdom at the University Press, Cambridge

A catalogue record for this publication is available from the British Library

ISBN 978-1107-01614-9 Hardback

Cambridge University Press has no responsibility for the persistence or accuracy of URLs for external or third-party internet websites referred to in this publication, and does not guarantee that any content on such websites is, or will remain, accurate or appropriate.

Cambridge University Press

978-1-107-01614-9 - Malliavin Calculus for Lévy Processes and Infinite-Dimensional Brownian
Motion: An Introduction

Horst Osswald

Frontmatter

[More information](#)

To
Ruth
Christine, Silas, Till
Fabian
and in memoriam
Horst

Cambridge University Press

978-1-107-01614-9 - Malliavin Calculus for Lévy Processes and Infinite-Dimensional Brownian
Motion: An Introduction

Horst Osswald

Frontmatter

[More information](#)

Contents

<i>Preface</i>	<i>page xv</i>
PART I THE FUNDAMENTAL PRINCIPLES	
1 Preliminaries	3
2 Martingales	9
2.1 Martingales and examples	9
2.2 Stopping times	12
2.3 The maximum inequality	13
2.4 Doob's inequality	14
2.5 The σ -algebra over the past of a stopping time	16
2.6 L^p -spaces of martingales and the quadratic variation norm	17
2.7 The supremum norm	19
2.8 Martingales of bounded mean oscillation	19
2.9 $(L^1_{\uparrow})'$ is BMO_2	21
2.10 $(L^1_{\sim})'$ is BMO_1	24
2.11 B–D–G inequalities for $p = 1$	28
2.12 The B–D–G inequalities for the conditional expectation for $p = 1$	30
2.13 The B–D–G inequalities	31
2.14 The B–D–G inequalities for special convex functions	32
Exercises	36
3 Fourier and Laplace transformations	37
3.1 Transformations of measures	37
3.2 Laplace characterization of $\mathcal{N}(0, \sigma)$ -distribution	38
3.3 Fourier and Laplace characterization of independence	39

Cambridge University Press

978-1-107-01614-9 - Malliavin Calculus for Lévy Processes and Infinite-Dimensional Brownian Motion: An Introduction

Horst Osswald

Frontmatter

[More information](#)

viii

Contents

3.4	Discrete Lévy processes and their representation	42
3.5	Martingale characterization of Brownian motion	45
	Exercises	46
4	Abstract Wiener–Fréchet spaces	50
4.1	Projective systems of measures and their limit	50
4.2	Gaussian measures in Hilbert spaces	52
4.3	Abstract Wiener spaces	54
4.4	Cylinder sets in Fréchet spaces generate the Borel sets	57
4.5	Cylinder sets in Fréchet space valued continuous functions	61
4.6	Tensor products	62
4.7	Bochner integrable functions	64
4.8	The Wiener measure on $C_{\mathbb{B}}$ is the centred Gaussian measure of variance 1	66
	Exercises	70
5	Two concepts of no-anticipation in time	71
5.1	Predictability and adaptedness	71
5.2	Approximations of the Dirac δ -function	73
5.3	Convolutions of adapted functions are adapted	75
5.4	Adaptedness is equivalent to predictability	76
5.5	The weak approximation property	77
5.6	Elementary facts about L^p -spaces	78
	Exercises	81
6	†Malliavin calculus on real sequences	82
6.1	Orthogonal polynomials	82
6.2	Integration	84
6.3	Iterated integrals	85
6.4	Chaos decomposition	86
6.5	Malliavin derivative and Skorokhod integral	88
6.6	The integral as a special case of the Skorokhod integral	89
6.7	The Clark–Ocone formula	90
6.8	Examples	91
	Exercises	94
7	Introduction to poly-saturated models of mathematics	95
7.1	Models of mathematics	96
7.2	The main theorem	102
	Exercises	105

Cambridge University Press

978-1-107-01614-9 - Malliavin Calculus for Lévy Processes and Infinite-Dimensional Brownian Motion: An Introduction

Horst Osswald

Frontmatter

[More information](#)*Contents*

ix

8	Extension of the real numbers and properties	107
8.1	${}^*\mathbb{R}$ as an ordered field	107
8.2	The * extension of the positive integers	107
8.3	Hyperfinite sets and summation in ${}^*\mathbb{R}$	109
8.4	The underspill and overspill principles	110
8.5	The infinitesimals	110
8.6	Limited and unlimited numbers in ${}^*\mathbb{R}$	111
8.7	The standard part map on limited numbers	112
	Exercises	113
9	Topology	115
9.1	Monads	115
9.2	Hausdorff spaces	116
9.3	Continuity	117
9.4	Compactness	117
9.5	Convergence	118
9.6	The standard part of an internal set of nearstandard points is compact	119
9.7	From S -continuous to continuous functions	120
9.8	Hyperfinite representation of the tensor product	121
9.9	† The Skorokhod topology	124
	Exercises	128
10	Measure and integration on Loeb spaces	130
10.1	The construction of Loeb measures	130
10.2	Loeb measures over Gaussian measures	133
10.3	Loeb measurable functions	135
10.4	On Loeb product spaces	137
10.5	Lebesgue measure as a counting measure	138
10.6	Adapted Loeb spaces	142
10.7	S -integrability and equivalent conditions	143
10.8	Bochner integrability and S -integrability	145
10.9	Integrable functions defined on $\mathbb{N}^n \times \Lambda \times [0, \infty]^m$	149
10.10	Standard part of the conditional expectation	153
10.11	Witnesses of S -integrability	155
10.12	Keisler's Fubini theorem	157
10.13	S -integrability of internal martingales	160
10.14	S -continuity of internal martingales	160
10.15	On symmetric functions	165
10.16	The standard part of internal martingales	166
	Exercises	170

Cambridge University Press

978-1-107-01614-9 - Malliavin Calculus for Lévy Processes and Infinite-Dimensional Brownian Motion: An Introduction

Horst Osswald

Frontmatter

[More information](#)

x

Contents

PART II AN INTRODUCTION TO FINITE- AND
INFINITE-DIMENSIONAL STOCHASTIC
ANALYSIS

Introduction	175
11 From finite- to infinite-dimensional Brownian motion	177
11.1 On the underlying probability space	177
11.2 The internal Brownian motion	179
11.3 \mathcal{S} -integrability of the internal Brownian motion	181
11.4 The \mathcal{S} -continuity of the internal Brownian motion	182
11.5 One-dimensional Brownian motion	182
11.6 Lévy's inequality	183
11.7 The final construction	186
11.8 The Wiener space	190
Exercises	194
12 The Itô integral for infinite-dimensional Brownian motion	196
12.1 The \mathcal{S} -continuity of the internal Itô integral	196
12.2 On the \mathcal{S} -square-integrability of the internal Itô integral	203
12.3 The standard Itô integral	204
12.4 On the integrability of the Itô integral	207
12.5 $\mathcal{W}_{C_{\mathbb{H}}}$ is generated by the Wiener integrals	208
12.6 \dagger The distribution of the Wiener integrals	209
Exercises	210
13 The iterated integral	211
13.1 The iterated integral with and without parameters	211
13.2 The product of an internal iterated integral and an internal Wiener integral	216
13.3 The continuity of the standard iterated integral process	218
13.4 The $\mathcal{W}_{C_{\mathbb{H}}}$ -measurability of the iterated Itô integral	219
13.5 $I_n^M(f)$ is a continuous version of the standard part of $I_n^M(F)$	221
13.6 Continuous versions of internal iterated integral processes	222
13.7 Kolmogorov's continuity criterion	224
Exercises	228

Cambridge University Press

978-1-107-01614-9 - Malliavin Calculus for Lévy Processes and Infinite-Dimensional Brownian Motion: An Introduction

Horst Osswald

Frontmatter

[More information](#)*Contents*

xi

14	† Infinite-dimensional Ornstein–Uhlenbeck processes	229
14.1	Ornstein–Uhlenbeck processes for shifts given by Hilbert–Schmidt operators	231
14.2	Ornstein–Uhlenbeck processes for shifts by scalars	239
	Exercises	246
15	Lindstrøm’s construction of standard Lévy processes from discrete ones	247
15.1	Exponential moments for processes with limited increments	248
15.2	Limited Lévy processes	251
15.3	Approximation of limited processes by processes with limited increments	256
15.4	Splitting infinitesimals	256
15.5	Standard Lévy processes	257
15.6	Lévy measure	260
15.7	The Lévy–Khinchine formula	264
15.8	Lévy triplets generate Lévy processes	265
15.9	Each Lévy process can be divided into its continuous and pure jump part	266
	Exercises	270
16	Stochastic integration for Lévy processes	271
16.1	Orthogonalization of the increments	271
16.2	From internal random walks to the standard Lévy integral	275
16.3	Iterated integrals	278
16.4	Multiple integrals	282
16.5	The σ -algebra generated by the Wiener–Lévy integrals	283
	Exercises	286
PART III MALLIAVIN CALCULUS		
	Introduction	291
17	Chaos decomposition	293
17.1	Admissible sequences	293
17.2	Chaos expansion	296
17.3	A lifting theorem for functionals in $L^2_{\mathcal{W}}(\widehat{\Gamma})$	299
17.4	Chaos for functions without moments	300
17.5	Computation of the kernels	300

17.6	The kernels of the product of Wiener functionals	303
	Exercises	304
18	The Malliavin derivative	306
18.1	The domain of the derivative	306
18.2	The Clark–Ocone formula	308
18.3	A lifting theorem for the derivative	309
18.4	The directional derivative	310
18.5	A commutation rule for derivative and limit	312
18.6	The domain of the Malliavin derivative is a Hilbert space with respect to the norm $\ \cdot\ _{1,2}$	313
18.7	The range of the Malliavin derivative is closed	314
18.8	A commutation rule for the directional derivative	315
18.9	Product and chain rules for the Malliavin derivative	315
	Exercises	318
19	The Skorokhod integral	319
19.1	Decomposition of processes	319
19.2	Malliavin derivative of processes	322
19.3	The domain of the Skorokhod integral	323
19.4	A lifting theorem for the integral	324
19.5	The Itô integral is a special case of the Skorokhod integral	325
	Exercises	327
20	The interplay between derivative and integral	328
20.1	The integral is the adjoint operator of the derivative	328
20.2	A Malliavin differentiable function multiplied by square-integrable deterministic functions is Skorokhod integrable	330
20.3	The duality between the domains of D and δ	332
20.4	$L^2_{\mathcal{W} \otimes \mathcal{L}^1}(\widehat{\Gamma \otimes v}, \mathbb{H})$ is the orthogonal sum of the range of D and the kernel of δ	333
20.5	Integration by parts	334
	Exercises	334
21	Skorokhod integral processes	335
21.1	The Skorokhod integral process operator	335
21.2	On continuous versions of Skorokhod integral processes	336
	Exercises	338

Cambridge University Press

978-1-107-01614-9 - Malliavin Calculus for Lévy Processes and Infinite-Dimensional Brownian Motion: An Introduction

Horst Osswald

Frontmatter

[More information](#)*Contents*

xiii

22	Girsanov transformations	339
22.1	From standard to internal shifts	341
22.2	The Jacobian determinant of the internal shift	342
22.3	Time-anticipating Girsanov transformations	343
22.4	Adapted Girsanov transformation	347
22.5	[†] Extension of abstract Wiener spaces	348
	Exercises	350
23	Malliavin calculus for Lévy processes	352
23.1	Chaos	352
23.2	Malliavin derivative	356
23.3	The Clark–Ocone formula	357
23.4	Skorokhod integral processes	358
23.5	Smooth representations	360
23.6	A commutation rule for derivative and limit	362
23.7	The product rule	362
23.8	The chain rule	366
23.9	Girsanov transformations	368
	Exercises	374
APPENDICES EXISTENCE OF POLY-SATURATED MODELS		
Appendix A.	Poly-saturated models	379
A.1	Weak models and models of mathematics	379
A.2	From weak models to models	380
A.3	Languages for models	381
A.4	Interpretation of the language	382
A.5	Models closed under definition	383
A.6	Elementary embeddings	384
A.7	Poly-saturated models	386
Appendix B.	The existence of poly-saturated models	388
B.1	From pre-models to models	388
B.2	Ultrapowers	390
B.3	Elementary chains and their elementary limits	393
B.4	Existence of poly-saturated models with the same properties as standard models	395
	<i>References</i>	398
	<i>Index</i>	404

Cambridge University Press

978-1-107-01614-9 - Malliavin Calculus for Lévy Processes and Infinite-Dimensional Brownian
Motion: An Introduction

Horst Osswald

Frontmatter

[More information](#)

Cambridge University Press

978-1-107-01614-9 - Malliavin Calculus for Lévy Processes and Infinite-Dimensional Brownian Motion: An Introduction

Horst Osswald

Frontmatter

[More information](#)

Preface

The aim of this book is to give a self-contained introduction to Malliavin calculus for Lévy processes $L: \Omega \times [0, \infty[\rightarrow \mathbb{B}$, where \mathbb{B} is a finite-dimensional Euclidean space or a separable Fréchet space, given by a countable sequence of semi-norms. We only take for granted that the reader has some knowledge of basic probability theory and functional analysis within the scope of excellent books on these fields, for example, Ash [5] or Billingsley [13] and Reed and Simon [98] or Rudin [101].

The most important Lévy processes are Brownian motion and Poisson processes, where Brownian motion is continuous and Poisson processes have jumps.

In Chapter 6 we will study Malliavin calculus for discrete stochastic processes $f: (\mathbb{R}^d)^{\mathbb{N}} \times \mathbb{N} \rightarrow \mathbb{R}^d$. The probability measure on $(\mathbb{R}^d)^{\mathbb{N}}$ is the product of a Borel probability measure on \mathbb{R}^d . For simplicity let us set $d = 1$; later on we accept $d = \infty$. In an application we obtain calculus for abstract Wiener spaces over l^2 , the space of square summable real sequences. By using suitable extensions of \mathbb{R} and \mathbb{N} , we obtain calculus for abstract Wiener spaces over arbitrary separable Hilbert spaces in the same manner, where we only identify two spaces if there exists a canonical, i.e., basis independent, isomorphic isometry between them.

In the short but very crucial Chapter 7 we extend \mathbb{R} and \mathbb{N} to ${}^*\mathbb{R}$ and ${}^*\mathbb{N}$ in such a way that the elements of ${}^*\mathbb{R}$ and ${}^*\mathbb{N}$ can be handled as though they were the usual real numbers and positive integers, respectively. In ${}^*\mathbb{N}$ there exist **infinitely large** positive integers H , which means that $n < H$ for all $n \in \mathbb{N}$. We take an infinitely large $H \in {}^*\mathbb{N}$ and use $T := \{\frac{i}{H} \mid i \in {}^*\mathbb{N}, i \leq H^2\}$ instead of $\frac{{}^*\mathbb{N}}{H}$. Then T is like a finite set and can be seen as an infinitely fine partition of $[0, \infty[$. It follows that there is no great difference between T and $[0, \infty[$. Our fixed sample space now is $\Omega_d := ({}^*\mathbb{R}^d)^T$. It only depends on the dimension d of the Lévy process we have in mind. If d is infinite, we can take Ω_ω , where ω is

Cambridge University Press

978-1-107-01614-9 - Malliavin Calculus for Lévy Processes and Infinite-Dimensional Brownian Motion: An Introduction

Horst Osswald

Frontmatter

[More information](#)

xvi

Preface

again an infinitely large number in ${}^*\mathbb{N}$. It turns out that each d, ∞ -dimensional Lévy process L lives on $\Omega := \Omega_d, \Omega_\omega$ (see Theorem 15.8.1 for $d = 1$ and Theorem 11.7.7 for ∞ -dimensional Brownian motion), i.e., L is a mapping from $\Omega \times [0, \infty[$ into \mathbb{R}^d , where \mathbb{R}^d is a Fréchet space in case $d = \infty$.

Moreover, since T is infinitely close to the continuous time line $[0, \infty[$, processes $f : \Omega \times [0, \infty[\rightarrow \mathbb{R}^d$ are infinitely close to processes $F : \Omega \times T \rightarrow {}^*\mathbb{R}^d$, where ${}^*\mathbb{R}^d = {}^*\mathbb{R}^\omega$ in the infinite-dimensional case. This relation ‘infinitely close’ will be studied and applied in the whole book in great detail.

I hope that this short Chapter 7 may help to achieve my most cherished objective to convince my gentle readers that there is no reason to fear model-theoretical reasoning in mathematics.

The choice of the sample space Ω implies that we can study finite- and infinite-dimensional Lévy processes simultaneously. Although Ω can be handled as though it were a finite-dimensional Euclidean space (even in the infinite-dimensional case), Ω is very rich. In particular, each right continuous function from $[0, \infty[$ into \mathbb{B} is a path of each Lévy process $L : \Omega \times [0, \infty[\rightarrow \mathbb{B}$. The proof of this fact is simple, but the result may be surprising and seems to be inconsistent.

This inconsistency disappears by observing that, in the case of Brownian motion, the set of non-continuous functions $f : [0, \infty[\rightarrow \mathbb{B}$ or the set of functions not starting in 0 is a nullset with respect to the image measure of the probability measure on Ω by L (it is the Wiener measure).

In the case of Poisson processes, the set of functions that are not increasing or fail to be counting functions or have only a finite range is a nullset.

In both cases the set of right continuous functions not having left-hand limits is also a nullset. It follows that we may assume that all Lévy processes L are almost surely surjective mappings from Ω onto the space D of **càdlàg** functions $f : [0, \infty[\rightarrow \mathbb{B}$, i.e., f is right continuous and has left-hand limits. Indeed, from each càdlàg function $f : [0, \infty[\rightarrow \mathbb{B}$ we can explicitly construct an $X \in \Omega$ with $L(X, \cdot) = f$.

One aim is to construct Brownian motion $b_{\mathbb{B}} : \Omega \times [0, \infty[\rightarrow \mathbb{B}$, where \mathbb{B} is a separable Fréchet space with metric d , generated by a sequence $(|\cdot|_i)_{i \in \mathbb{N}}$ of separating semi-norms $|\cdot|_i$ on \mathbb{B} . What is a \mathbb{B} -valued Brownian motion? According to the finite-dimensional situation $\mathbb{B} = \mathbb{R}^m$, it is required that the components are one-dimensional Brownian motions, running independently on orthonormal axes; axes are elements of an orthonormal basis of \mathbb{R}^m . A new question arises immediately: what does ‘orthogonality’ mean in infinite-dimensional Fréchet spaces \mathbb{B} ? Here is an answer: there exists a Hilbert space $\mathbb{H} \subseteq \mathbb{B}$ with norm, say $\|\cdot\|$, such that (\mathbb{B}, d) is the completion of (\mathbb{H}, d) and such that $\varphi \upharpoonright \mathbb{H}$ is continuous with respect to $\|\cdot\|$ for each φ in the topological dual \mathbb{B}' of \mathbb{B} . Then

\mathbb{B}' is a dense subspace of $\mathbb{H}' = \mathbb{H}$, with respect to $\|\cdot\|$ (see Section 4.3). So, orthogonality can be defined for all elements of \mathbb{B}' . Now $b_{\mathbb{B}} : \Omega \times [0, \infty[\rightarrow \mathbb{B}$ is a Brownian motion (BM), provided that $\varphi \circ b_{\mathbb{B}}$ is a one-dimensional BM for all $\varphi \in \mathbb{B}'$ with $\|\varphi\| = 1$, and $\varphi \circ b_{\mathbb{B}}$ and $\psi \circ b_{\mathbb{B}}$ are independent, provided that φ is orthogonal to ψ . So, axes of \mathbb{B} are orthogonal elements of the dual space of \mathbb{B} . Since orthogonal sets in \mathbb{R}^m can be identified with orthogonal sets in $(\mathbb{R}^m)'$, the notion of infinite-dimensional Brownian motion is literally a generalization of the finite-dimensional concept.

The Fréchet space \mathbb{B} is called an **abstract Wiener space** over the so-called **Cameron–Martin space** \mathbb{H} . A famous result, due to Leonhard Gross [41], tells us that each Gaussian measure on the algebra of cylinder sets of \mathbb{H} can be extended to a σ -additive measure $\gamma_{\mathbb{B}}$ on the Borel algebra of the described extension \mathbb{B} of \mathbb{H} . Here is a nice example: The space of real sequences, endowed with the topology of pointwise convergence, is an abstract Wiener–Fréchet space over l^2 .

There are many, quite different, abstract Wiener spaces over l^2 . However, it will be seen that for any two abstract Wiener spaces \mathbb{B} and \mathbb{D} over the same Hilbert space \mathbb{H} and for all $p \in [0, \infty]$ the L^p -spaces $L^p(\mathbb{B}, \gamma_{\mathbb{B}})$ and $L^p(\mathbb{D}, \gamma_{\mathbb{D}})$ can be identified, because there exists a canonical (i.e., basis independent) isomorphic isometry between them.

Our aim is to study:

- **in the finite-dimensional case:** Malliavin calculus on the space D of càdlàg functions, endowed with probability measures, generated by a large class of Lévy processes;
- **in the infinite-dimensional case:** Malliavin calculus on the space $C_{\mathbb{B}}$ of continuous functions from $[0, \infty]$ into \mathbb{B} , endowed with the Wiener measure. The reason why we take the space $C_{\mathbb{B}}$ instead of \mathbb{B} (both are Fréchet spaces) is the following: in analogy to the classical Wiener space we want to use the timeline $[0, \infty]$ in order to be able to define the notions ‘no time-anticipating’ and ‘Itô integral’. We replace in the classical Wiener space $C_{\mathbb{R}}$ the set \mathbb{R} of real numbers by separable Fréchet spaces \mathbb{B} .

Since our sample space Ω is finite-dimensional in a certain sense, Ω is much simpler to handle than the space D of càdlàg functions or the space $C_{\mathbb{B}}$. Therefore, we may work on Ω without any loss of generality. In addition, Ω is even richer than $C_{\mathbb{B}}$. Because of the choice of Ω , Malliavin calculus reduces to finite-dimensional analysis. In particular, the Malliavin derivative is the usual derivative (see Example 18.3.2).

Cambridge University Press

978-1-107-01614-9 - Malliavin Calculus for Lévy Processes and Infinite-Dimensional Brownian Motion: An Introduction

Horst Osswald

Frontmatter

[More information](#)

xviii

Preface

This book is an extension of a two-semester course on the Malliavin calculus, given in winter 2000/2001 and in summer 2001 at the University of Munich. It is organized as follows:

The results in Chapters 2, 3, 4 and 5 are well known and serve as the basis for the whole book.

In order to make preparations for the general Malliavin calculus we sketch a simple example in Chapter 6, following [88]. Although we work on \mathbb{N} and on powers of \mathbb{N} , Malliavin calculus is obtained for Poisson processes and Brownian motion with values in abstract Wiener spaces over l^2 . As we have already mentioned, techniques of Chapter 6 will be used later by enlarging \mathbb{N} to a rich set which can be treated as though it were a finite set.

Models of mathematics (so called poly-saturated models), in which finite extensions of \mathbb{N} exist, are introduced in Chapter 7.

Chapters 8 and 9 present some well-known applications of poly-saturated models to the real numbers and elementary topology.

Chapter 10 gives a detailed introduction to the well-established Loeb spaces with special regard to martingale theory.

Chapters 11, 12, 13 and 16 contain stochastic integration for infinite-dimensional Brownian motion and certain finite-dimensional Lévy processes, following [83], [84], [85], [89], [90], [91]. The results are partially an extension of results due to Cutland [23] and Cutland and Ng [24] for the special case of finite-dimensional Brownian motion.

In an application of Chapters 11, 12 and 13 we construct path-by-path continuous solutions to certain Langevin equations in infinite dimension, following [87] (see Chapter 14). Using this construction, one can easily see that each continuous function is a path of the solutions.

Following the work of Lindstrøm [67], we will show in Chapter 15 that each Lévy triplet can be satisfied by a Lévy process, which is infinitely close to a Lévy process, defined on a finite timeline, finite in the extended sense. Moreover, we will prove the well-known result that each Lévy process can be divided into a constant, a continuous Lévy martingale and a pure jump Lévy martingale.

Finally, Chapters 17 through 23 treat Malliavin calculus for infinite-dimensional Brownian motion and for a large class of finite-dimensional Lévy processes, following [83], [84], [85], [90], [89], [91], [92]. Again the results are partially extensions of results due to Cutland and Ng [24] for finite-dimensional Brownian motion.

In the appendices at the end of the book the reader can find a detailed proof of the existence of poly-saturated models of mathematics, following [86].

Cambridge University Press

978-1-107-01614-9 - Malliavin Calculus for Lévy Processes and Infinite-Dimensional Brownian Motion: An Introduction

Horst Osswald

Frontmatter

[More information](#)*Preface*

xix

In order to avoid technical difficulties, without being seriously less general, and in order to make the book easier to read, we only take two dimensions d , namely $d = 1$ and $d = \infty$.

We start with $d = \infty$, thus with Malliavin calculus on abstract Wiener spaces. The case $d = 1$ seems to be much simpler, but additional difficulties appear in connection with Lévy processes more general than Gaussian processes.

Each chapter ends with exercises, which we try to keep as close as possible to the subject of that chapter. A † at headings of chapters or sections indicates topics that may be omitted on first reading.

Acknowledgement: I am very grateful to Ralph Matthes for many helpful comments. I also express my sincere thanks to my colleagues and friends Josef Berger, Erwin Brüning, Cornelius Greither and Martin Schottenloher, who have read large parts of the manuscript and have sent me corrections, criticisms and many other useful comments. I wish to thank Cambridge University Press for their kindness and help in the production of the book. In particular, my thanks go to the copy-editor, Mairi Sutherland, for her careful reading of the manuscript and for many queries which led to an improvement of the text.

Cambridge University Press

978-1-107-01614-9 - Malliavin Calculus for Lévy Processes and Infinite-Dimensional Brownian
Motion: An Introduction

Horst Osswald

Frontmatter

[More information](#)
