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Malliavin Calculus for Lévy Processes and Infinite-Dimensional Brownian Motion

An Introduction

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Preface

The aim of this book is to give a self-contained introduction to Malliavin calculus for Lévy processes $L : \Omega \times [0, \infty[\to \mathbb{B}$, where $\mathbb{B}$ is a finite-dimensional Euclidean space or a separable Fréchet space, given by a countable sequence of semi-norms. We only take for granted that the reader has some knowledge of basic probability theory and functional analysis within the scope of excellent books on these fields, for example, Ash [5] or Billingsley [13] and Reed and Simon [98] or Rudin [101].

The most important Lévy processes are Brownian motion and Poisson processes, where Brownian motion is continuous and Poisson processes have jumps.

In Chapter 6 we will study Malliavin calculus for discrete stochastic processes $f : \left(\mathbb{R}^d\right)^{\mathbb{N}} \times \mathbb{N} \to \mathbb{R}^d$. The probability measure on $\left(\mathbb{R}^d\right)^{\mathbb{N}}$ is the product of a Borel probability measure on $\mathbb{R}^d$. For simplicity let us set $d = 1$; later on we accept $d = \infty$. In an application we obtain calculus for abstract Wiener spaces over l^2, the space of square summable real sequences. By using suitable extensions of $\mathbb{R}$ and $\mathbb{N}$, we obtain calculus for abstract Wiener spaces over arbitrary separable Hilbert spaces in the same manner, where we only identify two spaces if there exists a canonical, i.e., basis independent, isomorphic isometry between them.

In the short but very crucial Chapter 7 we extend $\mathbb{R}$ and $\mathbb{N}$ to ${}^*\mathbb{R}$ and ${}^*\mathbb{N}$ in such a way that the elements of ${}^*\mathbb{R}$ and ${}^*\mathbb{N}$ can be handled as though they were the usual real numbers and positive integers, respectively. In ${}^*\mathbb{N}$ there exist **infinitely large** positive integers H, which means that $n < H$ for all $n \in \mathbb{N}$. We take an infinitely large $H \in {}^*\mathbb{N}$ and use $T := \left\{ \frac{i}{H} \mid i \in {}^*\mathbb{N}, i \leq H^2 \right\}$ instead of $\frac{{}^*\mathbb{N}}{H}$. Then T is like a finite set and can be seen as an infinitely fine partition of $[0, \infty]$. It follows that there is no great difference between T and $[0, \infty[$. Our fixed sample space now is $\Omega_d := \left({}^*\mathbb{R}^d\right)^T$. It only depends on the dimension d of the Lévy process we have in mind. If d is infinite, we can take Ω_ω, where ω is again an infinitely large number in ${}^*\mathbb{N}$. It turns out that each d,∞-dimensional Lévy process L lives on $\Omega := \Omega_d,\ \Omega_\omega$ (see Theorem 15.8.1 for $d=1$ and Theorem 11.7.7 for ∞-dimensional Brownian motion), i.e., L is a mapping from $\Omega \times [0,\infty[$ into $\mathbb{R}^d$, where $\mathbb{R}^d$ is a Fréchet space in case $d=\infty$.

Moreover, since T is infinitely close to the continuous time line $[0,\infty[$, processes $f : \Omega \times [0,\infty[\to \mathbb{R}^d$ are infinitely close to processes $F : \Omega \times T \to {}^*\mathbb{R}^d$, where ${}^*\mathbb{R}^d = {}^*\mathbb{R}^\omega$ in the infinite-dimensional case. This relation 'infinitely close' will be studied and applied in the whole book in great detail.

I hope that this short Chapter 7 may help to achieve my most cherished objective to convince my gentle readers that there is no reason to fear model-theoretical reasoning in mathematics.

The choice of the sample space Ω implies that we can study finite- and infinite-dimensional Lévy processes simultaneously. Although Ω can be handled as though it were a finite-dimensional Euclidean space (even in the infinite-dimensional case), Ω is very rich. In particular, each right continuous function from $[0,\infty[$ into $\mathbb{B}$ is a path of each Lévy process $L : \Omega \times [0,\infty[\to \mathbb{B}$. The proof of this fact is simple, but the result may be surprising and seems to be inconsistent.

This inconsistency disappears by observing that, in the case of Brownian motion, the set of non-continuous functions $f : [0,\infty[\to \mathbb{B}$ or the set of functions not starting in 0 is a nullset with respect to the image measure of the probability measure on Ω by L (it is the Wiener measure).

In the case of Poisson processes, the set of functions that are not increasing or fail to be counting functions or have only a finite range is a nullset.

In both cases the set of right continuous functions not having left-hand limits is also a nullset. It follows that we may assume that all Lévy processes L are almost surely surjective mappings from Ω onto the space D of **càdlàg** functions $f : [0,\infty[\to \mathbb{B}$, i.e., f is right continuous and has left-hand limits. Indeed, from each càdlàg function $f : [0,\infty[\to \mathbb{B}$ we can explicitly construct an $X \in \Omega$ with $L(X,\cdot) = f$.

One aim is to construct Brownian motion $b_{\mathbb{B}} : \Omega \times [0,\infty[\to \mathbb{B}$, where $\mathbb{B}$ is a separable Fréchet space with metric d, generated by a sequence $(|\cdot|_i)_{i\in\mathbb{N}}$ of separating semi-norms $|\cdot|_i$ on $\mathbb{B}$. What is a $\mathbb{B}$-valued Brownian motion? According to the finite-dimensional situation $\mathbb{B} = \mathbb{R}^m$, it is required that the components are one-dimensional Brownian motions, running independently on orthonormal axes; axes are elements of an orthonormal basis of $\mathbb{R}^m$. A new question arises immediately: what does 'orthogonality' mean in infinite-dimensional Fréchet spaces $\mathbb{B}$? Here is an answer: there exists a Hilbert space $\mathbb{H} \subseteq \mathbb{B}$ with norm, say $\|\cdot\|$, such that $(\mathbb{B},d)$ is the completion of $(\mathbb{H},d)$ and such that $\varphi \restriction \mathbb{H}$ is continuous with respect to $\|\cdot\|$ for each φ in the topological dual $\mathbb{B}'$ of $\mathbb{B}$. Then

$\mathbb{B}'$ is a dense subspace of $\mathbb{H}' = \mathbb{H}$, with respect to $\|\cdot\|$ (see Section 4.3). So, orthogonality can be defined for all elements of $\mathbb{B}'$. Now $b_{\mathbb{B}} : \Omega \times [0, \infty[\to \mathbb{B}$ is a Brownian motion (BM), provided that $\varphi \circ b_{\mathbb{B}}$ is a one-dimensional BM for all $\varphi \in \mathbb{B}'$ with $\|\varphi\| = 1$, and $\varphi \circ b_{\mathbb{B}}$ and $\psi \circ b_{\mathbb{B}}$ are independent, provided that φ is orthogonal to ψ. So, axes of $\mathbb{B}$ are orthogonal elements of the dual space of $\mathbb{B}$. Since orthogonal sets in $\mathbb{R}^m$ can be identified with orthogonal sets in $(\mathbb{R}^m)'$, the notion of infinite-dimensional Brownian motion is literally a generalization of the finite-dimensional concept.

The Fréchet space $\mathbb{B}$ is called an **abstract Wiener space** over the so-called **Cameron–Martin space** $\mathbb{H}$. A famous result, due to Leonhard Gross [41], tells us that each Gaussian measure on the algebra of cylinder sets of $\mathbb{H}$ can be extended to a σ-additive measure $\gamma_{\mathbb{B}}$ on the Borel algebra of the described extension $\mathbb{B}$ of $\mathbb{H}$. Here is a nice example: The space of real sequences, endowed with the topology of pointwise convergence, is an abstract Wiener–Fréchet space over l^2.

There are many, quite different, abstract Wiener spaces over l^2. However, it will be seen that for any two abstract Wiener spaces $\mathbb{B}$ and $\mathbb{D}$ over the same Hilbert space $\mathbb{H}$ and for all $p \in [0, \infty]$ the L^p-spaces $L^p(\mathbb{B}, \gamma_{\mathbb{B}})$ and $L^p(\mathbb{D}, \gamma_{\mathbb{D}})$ can be identified, because there exists a canonical (i.e., basis independent) isomorphic isometry between them.

Our aim is to study:

- **in the finite-dimensional case**: Malliavin calculus on the space D of càdlàg functions, endowed with probability measures, generated by a large class of Lévy processes;
- **in the infinite-dimensional case**: Malliavin calculus on the space $C_{\mathbb{B}}$ of continuous functions from $[0, \infty]$ into $\mathbb{B}$, endowed with the Wiener measure. The reason why we take the space $C_{\mathbb{B}}$ instead of $\mathbb{B}$ (both are Fréchet spaces) is the following: in analogy to the classical Wiener space we want to use the timeline $[0, \infty[$ in order to be able to define the notions 'no time-anticipating' and 'Itô integral'. We replace in the classical Wiener space $C_{\mathbb{R}}$ the set $\mathbb{R}$ of real numbers by separable Fréchet spaces $\mathbb{B}$.

Since our sample space Ω is finite-dimensional in a certain sense, Ω is much simpler to handle than the space D of càdlàg functions or the space $C_{\mathbb{B}}$. Therefore, we may work on Ω without any loss of generality. In addition, Ω is even richer than $C_{\mathbb{B}}$. Because of the choice of Ω, Malliavin calculus reduces to finite-dimensional analysis. In particular, the Malliavin derivative is the usual derivative (see Example 18.3.2).

This book is an extension of a two-semester course on the Malliavin calculus, given in winter 2000/2001 and in summer 2001 at the University of Munich. It is organized as follows:

The results in Chapters 2, 3, 4 and 5 are well known and serve as the basis for the whole book.

In order to make preparations for the general Malliavin calculus we sketch a simple example in Chapter 6, following [88]. Although we work on $\mathbb{N}$ and on powers of $\mathbb{N}$, Malliavin calculus is obtained for Poisson processes and Brownian motion with values in abstract Wiener spaces over l^2. As we have already mentioned, techniques of Chapter 6 will be used later by enlarging $\mathbb{N}$ to a rich set which can be treated as though it were a finite set.

Models of mathematics (so called poly-saturated models), in which finite extensions of $\mathbb{N}$ exist, are introduced in Chapter 7.

Chapters 8 and 9 present some well-known applications of poly-saturated models to the real numbers and elementary topology.

Chapter 10 gives a detailed introduction to the well-established Loeb spaces with special regard to martingale theory.

Chapters 11, 12, 13 and 16 contain stochastic integration for infinite-dimensional Brownian motion and certain finite-dimensional Lévy processes, following [83], [84], [85], [89], [90], [91]. The results are partially an extension of results due to Cutland [23] and Cutland and Ng [24] for the special case of finite-dimensional Brownian motion.

In an application of Chapters 11, 12 and 13 we construct path-by-path continuous solutions to certain Langevin equations in infinite dimension, following [87] (see Chapter 14). Using this construction, one can easily see that each continuous function is a path of the solutions.

Following the work of Lindstrøm [67], we will show in Chapter 15 that each Lévy triplet can be satisfied by a Lévy process, which is infinitely close to a Lévy process, defined on a finite timeline, finite in the extended sense. Moreover, we will prove the well-known result that each Lévy process can be divided into a constant, a continuous Lévy martingale and a pure jump Lévy martingale.

Finally, Chapters 17 through 23 treat Malliavin calculus for infinite-dimensional Brownian motion and for a large class of finite-dimensional Lévy processes, following [83], [84], [85], [90], [89], [91], [92]. Again the results are partially extensions of results due to Cutland and Ng [24] for finite-dimensional Brownian motion.

In the appendices at the end of the book the reader can find a detailed proof of the existence of poly-saturated models of mathematics, following [86].

In order to avoid technical difficulties, without being seriously less general, and in order to make the book easier to read, we only take two dimensions d, namely $d = 1$ and $d = \infty$.

We start with $d = \infty$, thus with Malliavin calculus on abstract Wiener spaces. The case $d = 1$ seems to be much simpler, but additional difficulties appear in connection with Lévy processes more general than Gaussian processes.

Each chapter ends with exercises, which we try to keep as close as possible to the subject of that chapter. A † at headings of chapters or sections indicates topics that may be omitted on first reading.

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