Descriptive complexity theory establishes a connection between the computational complexity of algorithmic problems (the computational resources required to solve the problems) and their descriptive complexity (the language resources required to describe the problems).

This ground-breaking book approaches descriptive complexity from the angle of modern structural graph theory, specifically graph minor theory. It develops a ‘definable structure theory’ concerned with the logical definability of graph-theoretic concepts such as tree decompositions and embeddings.

The first part starts with an introduction to the background, from logic, complexity, and graph theory, and develops the theory up to first applications in descriptive complexity theory and graph isomorphism testing. It may serve as the basis for a graduate-level course. The second part is more advanced and mainly devoted to the proof of a single, previously unpublished theorem: properties of graphs with excluded minors are decidable in polynomial time if, and only if, they are definable in fixed-point logic with counting.

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CONTENTS

Preface ............................................................................................................. ix

Chapter 1. Introduction ............................................................................... 1
  1.1. Graph minor theory ................................................................. 2
  1.2. Treelike decompositions .................................................... 4
  1.3. Descriptive complexity theory ........................................... 6
  1.4. The graph isomorphism problem ........................................... 7
  1.5. The structure of this book ................................................... 8
  1.6. Bibliographical remarks ..................................................... 11

Part 1. The basic theory

Chapter 2. Background from graph theory and logic .......................... 14
  2.1. General notation ................................................................. 14
  2.2. Graphs and structures ....................................................... 15
  2.3. Logics ............................................................................... 22
  2.4. Transductions ................................................................. 32

Chapter 3. Descriptive complexity ...................................................... 40
  3.1. Logics capturing complexity classes ................................... 41
  3.2. Definable orders ................................................................. 51
  3.3. Definable canonisation ....................................................... 54
  3.4. Finite variable logics and pebble games .............................. 71
  3.5. Isomorphism testing and the Weisfeiler–Leman algorithm ... 79

Chapter 4. Treelike decompositions ................................................... 94
  4.1. Tree decompositions .......................................................... 94
  4.2. Treelike decompositions .................................................... 97
  4.3. Normalising treelike decompositions .................................. 104
  4.4. Tight decompositions ........................................................ 110
  4.5. Isomorphisms, homomorphisms, and bisimulations .......... 116
  4.6. Tree decompositions and treelike decompositions ........... 118
vi

Contents

CHAPTER 5. Definable decompositions .................................. 123
5.1. Decomposition schemes ........................................ 123
5.2. Normalising definable decompositions ...................... 126
5.3. Definable tight decompositions ............................ 128
5.4. Lifting definability ........................................ 129
5.5. Parametrised decomposition schemes ...................... 130
5.6. The transitivity lemma ..................................... 133

CHAPTER 6. Graphs of bounded tree width .................... 148
6.1. Defining bounded-width decompositions ................ 148
6.2. Defining bounded-width decompositions top-down .... 151

CHAPTER 7. Ordered treelike decompositions ............... 155
7.1. Definitions and basic results ............................. 155
7.2. Parametrised o-decomposition schemes ................ 160
7.3. Extension lemmas ....................................... 161
7.4. Canonisation via definable ordered treelike decompositions. 166

CHAPTER 8. 3-connected components ............................ 176
8.1. Decomposition into 2-connected components ........... 176
8.2. 2-separators of 2-connected graphs ....................... 179
8.3. Decomposition into 3-connected components ........... 182

CHAPTER 9. Graphs embeddable in a surface ..................... 189
9.1. Surfaces and embeddings of graphs ....................... 189
9.2. Angles .................................................. 204
9.3. Planar graphs .......................................... 211
9.4. Graphs on arbitrary surfaces ............................ 218

Part 2. Definable decompositions of graphs with excluded minors

CHAPTER 10. Quasi-4-connected components ...................... 232
10.1. Hinges .................................................. 232
10.2. Decomposition into quasi-4-connected components .... 254
10.3. The Q4C Lifting Lemma ................................ 264

CHAPTER 11. K5-minor-free graphs ................................. 272
11.1. Decompositions ........................................ 272
11.2. Definability .......................................... 275

CHAPTER 12. Completions of pre-decompositions ............ 277
12.1. Pre-decompositions and completions .................... 277
12.2. Ordered completions ................................ 280
12.3. Bounded-width completions ........................... 281
12.4. Derivations of pre-decompositions ..................... 285
CONTENTS vii

12.5. The finite extension lemma for ordered completions ........... 286
12.6. The Q4C Completion Lemma .................................. 289

CHAPTER 13. ALMOST PLANAR GRAPHS ............................. 301
13.1. Relaxations of planarity ...................................... 302
13.2. Central vertices ........................................... 308
13.3. Defining the central faces ................................... 312
13.4. Centres and skeletons ....................................... 345
13.5. Decomposing almost planar graphs and their minors ........... 349

CHAPTER 14. ALMOST PLANAR COMPLETIONS ...................... 361
14.1. From almost planar to ordered completions .................... 361
14.2. Grids ..................................................... 363
14.3. Supercentre and superskeleton ................................ 378
14.4. The completion theorem for quasi-4-connected graphs ........ 380
14.5. MAP_p-star completions .................................... 385
14.6. Proof of the Almost Planar Completion Theorem 14.1.3 ..... 390

CHAPTER 15. ALMOST-EMBEDDABLE GRAPHS ....................... 393
15.1. Arrangements in a surface ................................... 393
15.2. Shortest path systems ....................................... 407
15.3. Simplifying and safe subgraphs ................................ 411
15.4. Patches ................................................. 413
15.5. Belts ..................................................... 426

CHAPTER 16. DECOMPOSITIONS OF ALMOST-EMBEDDABLE GRAPHS ... 438
16.1. The Combination Lemma ..................................... 438
16.2. The Last Extension Lemma ................................... 445
16.3. Decomposing almost-embeddable graphs and their minors .. 470
16.4. Almost-embeddable completions ................................ 476

CHAPTER 17. GRAPHS WITH EXCLUDED MINORS .................. 487
17.1. The structure of graphs with excluded minors ................. 487
17.2. The main theorem ......................................... 493

CHAPTER 18. BITS AND PIECES .................................... 502
18.1. From graphs to relational structures .......................... 502
18.2. Lifting canonisations ....................................... 504
18.3. Invariant decompositions and canonisation .................... 511
18.4. Directions for further research ................................ 513

APPENDIX A. ROBERTSON AND SEYMOUR’S VERSION OF THE LOCAL 
STRUCTURE THEOREM ........................................... 518

REFERENCES ..................................................... 523
## CONTENTS

<table>
<thead>
<tr>
<th>SYMBOL INDEX</th>
<th>INDEX</th>
</tr>
</thead>
<tbody>
<tr>
<td>531</td>
<td>535</td>
</tr>
</tbody>
</table>
PREFACE

This monograph evolved around the proof of a single theorem: fixed-point logic with counting captures polynomial time on all graph classes with excluded minors. The proof of this theorem heavily relies on structural graph theory, and the core question that needs to be addressed is how to make graph-theoretic concepts definable in logic. As many of those graph-theoretic concepts, for example, tree decompositions, are not invariant under isomorphisms and as isomorphism invariance is a prerequisite for being definable, the graph theory needs to be adapted. This leads to the definable graph structure theory presented in this monograph.

I started to work on this topic in 1997, a few years after I completed my PhD. At the time, I was mainly interested in finite model theory and especially in the main open problem of the area: the question of whether there is a logic that captures polynomial time. The results I had proved at the time were mostly “negative”: counterexamples to nice conjecture and inexpressibility results, usually involving the construction of very complicated graphs and combinatorial structures. I felt a certain desire to prove a “positive” result for once, so I started to look at simpler structures, in the naive hope that on such structures the complicated counterexamples could be avoided and everything would work out nicely. To cut a long story short: it did, though only after a few complications and learning a lot of graph theory.

Jörg Flum encouraged me to present the material in a book rather than a series of technical papers, and I think this was a good idea. This book has greatly improved through the discussions I had with and comments and corrections I received from my colleagues. I am very grateful to all of them! In particular, I would like to thank Achim Blumensath, Reinhard Diestel, Jörg Flum, Frederik Harwath, Neil Immerman, Skip Jordan, Stephan Kreutzer, Martin Otto, Pascal Schweitzer, Wolfgang Thomas.

ix