Cambridge University Press
978-1-107-01345-2 — Black Holes in Higher Dimensions
Edited by Gary T. Horowitz
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Part I

Introduction

1

Black holes in four dimensions

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In this chapter we briefly review black holes in four spacetime dimensions. (For more details see any standard textbook in general relativity such as [1, 2] and for a guide to the extensive literature see [3, 4].) We begin with some explicit solutions and then discuss their general properties. This will set the stage for our exploration of higher-dimensional black holes in the rest of the book.

1.1 Schwarzschild solution

Within months of Einstein's final formulation of general relativity in 1915, Schwarzschild found an exact solution given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{1.1}$$

where $d\Omega^2$ is shorthand for $d\theta^2 + \sin^2\theta \, d\phi^2$. The parameter M represents the total mass in units with $G = 1 = c$. This metric solves Einstein's equation with $T_{\mu\nu} = 0$, so it is called a vacuum solution. Since the metric is static and spherically symmetric, it was originally interpreted as describing the geometry outside a static spherical star.

The metric (1.1) clearly has a problem at $r = 2M$. For ordinary stars this is a few kilometers, which is much smaller than the radius of the star, so this problem can be ignored. However, it is known that after a star uses up its nuclear fuel it undergoes gravitational collapse. A sufficiently massive star will continue to collapse to essentially zero volume. If the star is spherically symmetric, the geometry outside is given by the Schwarzschild solution. So in this case, we can no longer ignore the problem at $r = 2M$.

Black Holes in Higher Dimensions, ed. G. Horowitz. Published by Cambridge University Press.

In general relativity, one must distinguish between curvature singularities and coordinate singularities. As its name suggests, a curvature singularity is a genuine singularity in spacetime at which the curvature diverges. In contrast, a coordinate singularity is a place where the curvature is perfectly fine but the metric components diverge owing to a bad choice of coordinates. If one computes the square of the Riemann tensor, $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$, for the Schwarzschild solution, one finds that it is proportional to M^2/r^6. This suggests that $r = 2M$ is just a coordinate singularity (and $r = 0$ is a real curvature singularity).

To find good coordinates across $r = 2M$ it is convenient to base them on the motion of physical particles. One possibility is to use ingoing radial photons. These satisfy

$$v_0 - t = \int \left(1 - \frac{2M}{r}\right)^{-1} dr = r + 2M \ln(r - 2M) \equiv r_* \tag{1.2}$$

where v_0 is a constant labeling the different ingoing radial photons. Setting $v = t + r_*$ and using v as a new time coordinate, the Schwarzschild metric takes the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2dvdr + r^2 d\Omega^2 \,. \tag{1.3}$$

These are called ingoing Eddington–Finkelstein coordinates. The metric is now nonsingular at $r = 2M$. The fact that $g_{vv} = 0$ there just means that the $\partial/\partial v$ Killing field changes from being timelike for $r > 2M$ to spacelike for $r < 2M$. The metric remains invertible and does not change signature. It is clear from (1.2) that $t \to \infty$ at $r = 2M$, which is why there is a coordinate singularity in the original form of the Schwarzschild solution.

By construction, ingoing light rays follow curves of constant v. Outgoing light rays satisfy

$$\frac{dr}{dv} = \frac{1}{2}\left(1 - \frac{2M}{r}\right) \,. \tag{1.4}$$

For $r > 2M$ the right-hand side is positive, so outgoing light rays move to larger values of r as expected. For $r = 2M$ the right-hand side vanishes, so these are outgoing light rays. For $r < 2M$ the right-hand side is negative, so even outgoing light rays are dragged to smaller radii. This means that any observer following a timelike worldline must decrease r and eventually hit the singularity. The surface $r = 2M$ is called the *event horizon* and the entire region $r \leq 2M$ is a *black hole*.

Surfaces of constant $r < 2M$ are spacelike. Thus, the singularity at $r = 0$ is also spacelike. In other words, the singularity is not at a particular location in space, but rather at a particular time. Inside the black hole, the singularity is in your future and cannot be avoided.

If we do not include matter then (1.3) does not cover the entire spacetime, even if we let $-\infty < v < \infty$, $0 < r < \infty$. One can show that $e^{v/4M}$ is an affine parameter along the outgoing null geodesics, so the affine parameter never reaches minus infinity in (1.3). Similarly, setting $u = t - r_*$, one can show that $e^{-u/4M}$ is an affine parameter along ingoing null geodesics. To construct the maximally extended spacetime, we introduce Kruskal coordinates X, T based on these affine parameters:

$$X = \frac{1}{2}(e^{v/4M} + e^{-u/4M}) = (r - 2M)^{1/2} e^{r/4M} \cosh \frac{t}{4M} \,,$$
$$T = \frac{1}{2}(e^{v/4M} - e^{-u/4M}) = (r - 2M)^{1/2} e^{r/4M} \sinh \frac{t}{4M} \,. \tag{1.5}$$

The Schwarzschild metric becomes

$$ds^2 = \frac{32M^3}{r} e^{-r/2M}(-dT^2 + dX^2) + r^2 d\Omega^2 \tag{1.6}$$

where r is now a function of T and X defined implicitly by

$$X^2 - T^2 = (r - 2M) e^{r/2M} \,. \tag{1.7}$$

The first thing to note about these new coordinates is that each surface of constant r is represented twice in the (X, T)-plane. There are two hyperbolas with the same value of r. In particular, there are two $r = 0$ singularities (one with $T > 0$ and one with $T < 0$), two asymptotic regions (one with $X \gg 0$ and one with $X \ll 0$), and two event horizons ($T = X$ and $T = -X$).

A convenient way to represent the causal structure of a spacetime is via a Penrose diagram. This is obtained upon conformally rescaling the metric $\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$ by a function that goes to zero at infinity, so that infinity is brought to a finite distance in the unphysical metric $\tilde{g}_{\mu\nu}$. If one does this for Minkowski spacetime, one finds that the boundary at infinity consists of two null surfaces, $\mathcal{I}^{\pm}$, called future and past null infinity, and three points representing spatial infinity and future and past timelike infinity. The conformal rescaling does not change causal relations. In a Penrose diagram, light cones are drawn at 45 degrees just as in Minkowski spacetime, so these causal relations are easy to see.

After conformally rescaling the metric (1.6) to bring infinity to a finite distance, we get the Penrose diagram shown in Fig. 1.1. Each point in this two-dimensional figure represents a two-sphere of radius r. Region I is the original asymptotically flat region and region II is the black hole. The ingoing Eddington–Finkelstein coordinates cover regions I and II. Region III is the time reverse of a black hole,

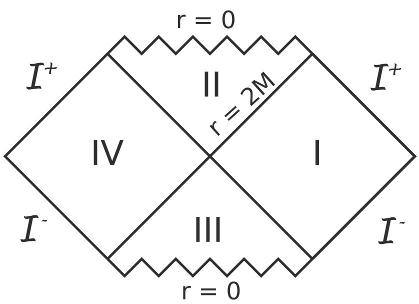

Figure 1.1 Penrose diagram of the maximally extended Schwarzschild solution. There are two asymptotically flat regions (I and IV), a black hole (II), and a white hole (III).

called a white hole. Material can come out of a white hole, but nothing can fall in. Region IV is another asymptotically flat region of spacetime similar to region I. However, it is clear that these two regions are causally disconnected. There is no way for someone in region I to communicate with someone in region IV. A spacelike surface stretching from one asymptotically flat region to the other has the geometry of a wormhole.

To summarize, we have seen that the simple-looking Schwarzschild solution (1.1) is full of surprises. Although originally viewed as the geometry outside a spherical star, it also describes a black hole, a white hole, and a second asymptotic region causally disconnected from the original one. However, one should keep in mind that the maximally extended vacuum spacetime is not very physical. A black hole that forms from a collapsing star does not have regions III or IV.

1.2 Reissner–Nordström solution

A few years after Schwarzschild found his solution, the generalization to the charged case was found by Reissner and Nordström. The metric takes a form similar to (1.1):

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2 \qquad (1.8)$$

where M is again the mass and Q is the charge. For an electrically charged black hole, the only nonzero component of the Maxwell field is $F_{rt} = Q/r^2$. For a magnetically charged black hole, the only nonzero component is $F_{\theta\phi} = Q \sin\theta$. This solves the Einstein–Maxwell equations, which follow from the action

$$S = \int d^4x \sqrt{-g}(R - F_{\mu\nu}F^{\mu\nu}) \, . \qquad (1.9)$$

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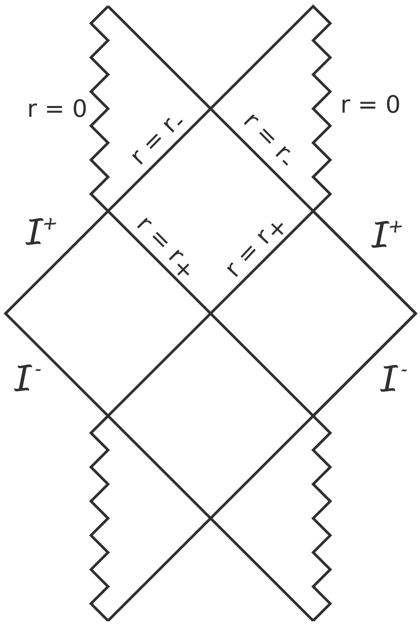

Figure 1.2 The Penrose diagram for the maximally extended Reissner–Nordström solution with $Q < M$. The diagram repeats infinitely often in the vertical direction.

There are three cases to consider, depending on the relative size of M and Q (we will assume $Q > 0$). One might wonder how we can compare mass and charge since they seem to have different units. The answer is to write everything in Planck units. Setting $\hbar = 1$ (as well as $G = c = 1$), the fundamental unit of electric charge is given by $e^2 = 1/137$, so $e \sim (1/10)m_{\text{Planck}} \sim 10^{-6}$ gm.

We consider first the case $M > Q$. It is convenient to write

$$1 - \frac{2M}{r} + \frac{Q^2}{r^2} = \frac{(r - r_+)(r - r_-)}{r^2} \tag{1.10}$$

where $r_\pm = M \pm \sqrt{M^2 - Q^2}$. The metric components are singular at $r = r_\pm$ and $r = 0$, but only $r = 0$ is a curvature singularity. The surfaces of constant r are timelike for $r < r_-$ so, unlike the Schwarzschild solution, the singularity is timelike. In other words, here the singularity occurs at a location in space and not in time, so it can be avoided. In fact this solution is timelike geodesically complete, so no freely falling observer hits the singularity. The complete spacetime is shown in Fig. 1.2. Note that an arbitrarily small charge Q causes a qualitative change in the causal structure of the solution.

The surface $r = r_+$ is the event horizon and the surface $r = r_-$ is the inner horizon or Cauchy horizon. The Cauchy horizon marks the limit where initial data on a spacelike surface stretching from one asymptotic region to the other has a unique evolution. Past this horizon the evolution can be modified by unknown boundary conditions at the singularity.

The inner horizon has been shown to be unstable: the slightest perturbation causes the curvature to diverge. To see why, notice that an observer crossing the inner horizon can look back and see the entire future of the asymptotically flat spacetime he or she left behind. Consider a star that is radiating at a fixed distance from the black hole. It will send radiation across the event horizon that gets blueshifted and builds up on the inner horizon. If the star radiates forever, an infinite amount of radiation piles up near $r = r_-$. Of course stars do not radiate forever, but it has been shown that any perturbation outside the horizon causes enough radiation to fall into the black hole for the curvature to diverge at the inner horizon [5, 6]. Interestingly, the singularity that forms is initially rather weak. It is null (not spacelike) and the metric is continuous (but not differentiable) across it. This means that even though the tidal forces diverge, the total tidal distortion remains finite. Whether someone could actually survive a journey across the inner horizon is not clear. There is good evidence that the aforementioned singularity eventually turns into a much stronger spacelike curvature singularity.

Since there is no charged matter, the charge enclosed inside a sphere of radius r is independent of r. Hence one can view the charge as residing at the singularity. The two singularities actually have charges of opposite sign. The reason is that the electric field seen by a family of observers is a spacelike vector pointing from left to right everywhere in Fig. 1.2. The singularity on the left has charge $Q > 0$, and the one on the right has charge $-Q$.

The case $M = Q$ is called the extremal limit. This is the maximum charge that one can put on a black hole of mass M. In this case $r_+ = r_- = Q$, so the spacetime has a single horizon that is degenerate, i.e., g_{tt} has a double zero. The metric takes the form

$$ds^2 = -\left(1 - \frac{Q}{r}\right)^2 dt^2 + \left(1 - \frac{Q}{r}\right)^{-2} dr^2 + r^2 d\Omega^2 . \tag{1.11}$$

(Recall that we are assuming charges are positive.) The causal structure is shown in Fig. 1.3.

The proper distance to the horizon from any point $r > Q$ along a surface of constant t is infinite. This may lead you to suspect that the black hole has receded to an infinite distance, but it has not: infalling observers reach the horizon in finite proper time. Similarly, radial null geodesics reach the horizon in finite affine parameter. Letting $\rho = r - Q$, the solution becomes

$$ds^2 = -h^{-2}(\rho)dt^2 + h^2(\rho)(d\rho^2 + \rho^2 d\Omega^2), \quad A_t = h^{-1} , \tag{1.12}$$

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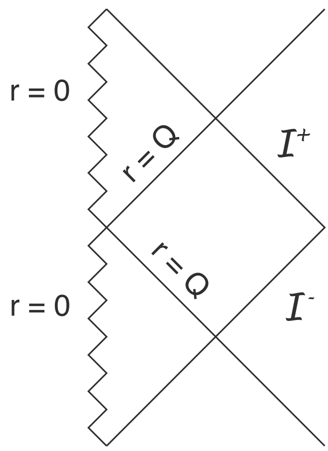

Figure 1.3 The Penrose diagram for the extreme charged black hole. The diagram repeats infinitely often in the vertical direction.

where $h = 1 + Q/\rho$. The horizon is now at $\rho = 0$. In fact, the entire set of Einstein–Maxwell equations reduce to a simple (flat-space!) Laplace equation on h: $\partial^2 h = 0$. Since this is a linear equation, one can superpose solutions. For example, introducing Cartesian coordinates $\vec{x}$ instead of the spherical coordinates ρ, θ, ϕ, the solution

$$h(\vec{x}) = 1 + \sum \frac{Q_i}{|\vec{x} - \vec{x}_i|} \tag{1.13}$$

describes a static collection of extremal black holes with charges Q_i and positions $\vec{x}_i$. There is no force between extremal black holes since the gravitational attraction is exactly balanced by the Coulomb repulsion. These are known as the Majumdar–Papapetrou solutions.

The fact that the proper distance to the horizon is infinite on a surface of constant t allows one to extract a limiting geometry near the horizon. Starting with (1.12) with $h = 1 + Q/\rho$, write $\rho = \epsilon\tilde{\rho}$, $t = \tilde{t}Q^2/\epsilon$, and take the limit $\epsilon \to 0$ with the tilded coordinates held fixed. The metric becomes

$$ds^2 = Q^2\left(-\tilde{\rho}^2 d\tilde{t}^2 + \frac{d\tilde{\rho}^2}{\tilde{\rho}^2} + d\Omega^2\right) . \tag{1.14}$$

This is the product of a two-sphere and a two-dimensional spacetime of constant negative curvature called the anti-de Sitter spacetime (AdS_2). Both these spaces have radius of curvature Q. This near-horizon geometry has more symmetry than the original spacetime. The time-translation symmetry is enhanced to $SO(2, 1)$,

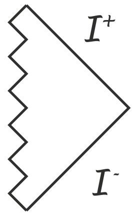

Figure 1.4 The Penrose diagram for the $Q > M$ Reissner–Nordström solution. This is also the Penrose diagram for the Schwarzschild solution with $M < 0$.

which is the symmetry group of AdS_2. We will talk more about anti-de Sitter spacetime in section 1.4, and extremal black holes will be discussed further in Chapter 11.

Finally, the case $Q > M$ does not describe a black hole at all (see Fig. 1.4). Here $g_{tt} < 0$ everywhere, so $r = 0$ is a timelike singularity that is naked, i.e., visible to distant observers. Can one form such a singularity from smooth initial conditions? It is easy to construct a ball of charged dust with $Q > M$ (recall that the fundamental unit of electric charge corresponds to 10^{-6} gm), but this will not collapse to $r = 0$. The Coulomb repulsion exceeds the gravitational attraction so it would take an infinite amount of energy to compress it to zero volume. We will discuss the possibility of forming naked singularities further in section 1.5.

1.3 Kerr solution

Although the Schwarzschild solution was found just months after the discovery of general relativity and the Reissner–Nordström solution was found a couple years later, it took almost 50 years before the analogous solution for a rotating black hole was found by Kerr. One reason for the delay was that the metric is considerably more complicated:

$$ds^2 = -\left(1 - \frac{2Mr}{\Sigma}\right) dt^2 - \frac{4Mar\sin^2\theta}{\Sigma} dt d\phi + \frac{\Sigma}{\Delta} dr^2 + \Sigma d\theta^2 + \left[\frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma}\right] \sin^2\theta \, d\phi^2 \,, \tag{1.15}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta, \qquad \Delta = r^2 - 2Mr + a^2 \,. \tag{1.16}$$

This metric solves the vacuum Einstein equation; it depends on two parameters, the mass M and the angular momentum $J = Ma$. Setting $a = 0$ yields the

Schwarzschild metric and setting $M = 0$ yields flat spacetime in unusual coordinates. The metric is independent of t and ϕ, so the spacetime is stationary and axisymmetric. It is not static, like the Schwarzschild metric, since it is not invariant under $t \to -t$, but it is invariant under simultaneous reflection in t and ϕ.

A key difference between this metric and those we have studied so far is that $g_{t\phi} \neq 0$. This is true for all rotating objects and has an important consequence. Consider an observer whose four-momentum is orthogonal to a constant-t surface. This observer is "at rest" with respect to the surface of constant t and has zero angular momentum (since $L = P_\mu(\partial/\partial\phi)^\mu$ and the rotational Killing field is tangent to the surface of constant t). Nevertheless, the observer must be rotating with respect to infinity. This can be seen as follows. The observer's four-momentum takes the form $P^\mu \propto (\partial/\partial t)^\mu + \Omega(\partial/\partial\phi)^\mu$, where Ω can be determined from $0 = P_\mu(\partial/\partial\phi)^\mu \propto g_{t\phi} + \Omega g_{\phi\phi}$. Since we also know that $P^\mu = \dot{t}(\partial/\partial t)^\mu + \dot{\phi}(\partial/\partial\phi)^\mu$, the coordinate angular velocity is nonzero:

$$\frac{d\phi}{dt} = \Omega = -\frac{g_{t\phi}}{g_{\phi\phi}} = \frac{2Mra}{(r^2 + a^2)^2 - \Delta a^2 \sin^2\theta} \,. \tag{1.17}$$

The fact that observers locally at rest must rotate with respect to infinity can be interpreted as a "dragging of inertial frames". This is a property not just of rotating black holes but of any rotating body in general relativity.

The metric components in (1.15) diverge when $\Sigma = 0$ or when $\Delta = 0$. Only the first is a true curvature singularity. Note that $\Sigma = 0$ implies $r = 0$ and $\theta = \pi/2$. Normally $r = 0$ is a point, and the values of the angular coordinates do not matter. The fact that the singularity only occurs when $r = 0$ and $\theta = \pi/2$ shows that these are not ordinary spherical coordinates. In terms of good coordinates, it turns out that constant-r surfaces (for small r) are ellipsoids that degenerate into a disk of proper radius a when $r = 0$. Surfaces of constant θ are hyperboloids that meet the ellipsoids orthogonally. The angle θ acts like a radial coordinate inside the disk $r = 0$. As a result, the singularity at $r = 0$ and $\theta = \pi/2$ is a ring. If one goes through the center of the ring, one finds another asymptotically flat region of spacetime whose metric is again given by (1.15) but now with $r < 0$.

In close analogy with the situation for the Reissner–Nordström metric, $\Delta = 0$ is a coordinate singularity. There are again three cases, depending on the ratio a/M. We discuss first the physically most interesting choice, $a < M$. In this case Δ has two roots, $r_\pm = M \pm \sqrt{M^2 - a^2}$. The larger root, r_+, is the event horizon and the smaller root, r_-, is an inner horizon. Since the spacetime is not spherically symmetric, one cannot represent the causal structure completely by a two-dimensional Penrose diagram. However, if one looks at the geometry along a radial line in the equatorial plane, the causal structure is similar to that in Fig. 1.2. If you move