1 Mathematics

As scientists and engineers, we make sense of the world around us through observation and experimentation. Using mathematics, we attempt to describe our observations and make useful predictions based on these observations. For example, a simple experimental observation that the distance traversed by an object traveling at a constant velocity is linearly related to both the velocity and the time can be formalized using the relation, $\boldsymbol{d} = \boldsymbol{v}\, t$, where $\boldsymbol{d}$ is the distance vector, $\boldsymbol{v}$ is the velocity vector, and t is the time. The distance, velocity, and time are physical quantities that can be measured or controlled. Physical quantities such as distance, velocity, and time are represented mathematically as tensors. A scalar, for example, is a zeroth-order tensor. Only a magnitude is required to specify the value of a zeroth-order tensor. In our previous example, time is such a quantity. If you are told that the duration of an event was 3 seconds, you need no other information to fully characterize this physical quantity. Velocity, on the other hand, requires both a magnitude and a direction to specify its meaning. The velocity would be represented using a first-order tensor, also known as a vector. The internal stress in a material is a second-order tensor, which requires a magnitude and two directions to specify its value. You may recognize that the two required directions are the normal of the surface on which the stress acts and the direction of the traction vector on this surface. Tensors of higher order require additional information to specify their physical meaning. In this chapter, we will review the basic tensor algebra and tensor calculus that will be used in the formulation of continuum representations.

1.1 Vectors

A first-order tensor, also known as a vector, is used to represent a physical quantity whose representation requires both direction and magnitude. However, additional requirements must be satisfied. First, two vectors must add according to the parallelogram rule. Second, if a vector is defined within a given reference frame, and a second rotated reference frame is defined, it must be possible to express the components of a vector in one reference frame in terms of the components within another reference frame.

Whereas the physical meaning of a vector, such as the velocity of a car, is independent of coordinate system, the components of a vector are not. If we define a set of ***orthonormal basis vectors***, $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$, we can express a vector as a linear combination of the basis vectors such that

$$\boldsymbol{a} = a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3,$$

where a_1, a_2, a_3 are scalars representing the components of the vector in the $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$ directions respectively. The ***magnitude*** of a vector, $|\boldsymbol{a}|$, is a measure of the length of a vector and is defined as

$$|\boldsymbol{a}| = \sqrt{\left(a_1^2 + a_2^2 + a_3^2\right)}.$$

It is often necessary to compare the relative size of two physical quantities whether they be scalars, vectors, or a general nth-order tensor. In each case, we may compare the ***norm*** of the two tensors. The norm of a scalar is equal to the absolute value of the scalar, whereas the norm of a vector, denoted as $\|\boldsymbol{a}\|$, is equal to its magnitude. Both the magnitude and the norm of a vector are zero if and only if each of the components of the vector is zero.

Whereas magnitude specifies the size of the vector, the direction of the vector may be represented by a ***unit vector***, $\hat{\boldsymbol{a}}$, parallel to the original vector, $\boldsymbol{a}$, such that

$$\hat{\boldsymbol{a}} = \frac{\boldsymbol{a}}{|\boldsymbol{a}|}.$$

This unit vector captures the directional information contained within the vector but discards the magnitude. The magnitude of any unit vector is equal to one. If two vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$, are parallel, one vector can be written as a scalar, α, times the other vector,

$$\boldsymbol{a} = \alpha\boldsymbol{b}.$$

Vector and tensor equations can become quite complicated. It is often possible to use ***index notation*** to simplify and manipulate the representation of vector or tensor equations. Let us begin with the assumption that we are modeling physical quantities in a three-dimensional space that is spanned by the orthonormal basis vectors, $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$. In order to write a vector equation in index notation, we introduce an index, i, which in this case is a variable that can assume the value of 1, 2, or 3. The representation of a vector as a linear combination of the basis vectors can be written in the compact form,

$$\boldsymbol{a} = a_1\boldsymbol{e}_1 + a_2\boldsymbol{e}_2 + a_3\boldsymbol{e}_3 = \sum_{\boldsymbol{i=1}}^{\boldsymbol{3}} a_i\boldsymbol{e}_i.$$

The summation from 1 to 3 over a repeated index is quite common and may be represented in a more compact form using the ***abbreviated summation convention*** which is also termed ***Einstein notation*** as

$$\boldsymbol{a} = a_i\boldsymbol{e}_i. \tag{1.1}$$

The abbreviated summation convention is implied if and only if an index appears exactly twice within the same term of an equation.

The sum of two vectors, $\boldsymbol{b}$ and $\boldsymbol{c}$, is equal to a vector such that

$$\boldsymbol{a} = \boldsymbol{b} + \boldsymbol{c}.$$

The addition of two vectors is both communitive, $\boldsymbol{b}+\boldsymbol{c}=\boldsymbol{c}+\boldsymbol{b}$, and consistent with the parallogram rule. The components of the vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ parallel to the same basis vector can be added. Vector addition can be written in terms of components such that

$$a_1\boldsymbol{e}_1+a_2\boldsymbol{e}_2+a_3\boldsymbol{e}_3=(b_1+c_1)\boldsymbol{e}_1+(b_2+c_2)\boldsymbol{e}_2+(b_3+c_3)\boldsymbol{e}_3.$$

This gives three separate equations for the components of the vector $\boldsymbol{a}$,

$$a_1=b_1+c_1,$$
$$a_2=b_2+c_2,$$
$$a_3=b_3+c_3.$$

In index notation, this set of three equations is represented as

$$a_i=b_i+c_i,$$

where i can take on a value of 1, 2, or 3. The subscript i, termed a ***free index***, appears exactly once in each of the terms in the equation. In contrast, the subscript i, appears twice in the right term in Equation (1.1). In that equation, the subscript is termed a ***dummy index*** which signifies a summation from 1 to 3 over the repeated indices.

The scalar valued ***dot product***, also known as a ***scalar product*** or ***inner product***, of two vectors is defined as

$$\boldsymbol{a}\cdot\boldsymbol{b}=|\boldsymbol{a}||\boldsymbol{b}|\cos\theta_{\boldsymbol{ab}}=a_1b_1+a_2b_2+a_3b_3=a_ib_i,$$

where $\theta_{\boldsymbol{ab}}$ is the angle between the two vectors. There are no free indices in this equation, but there is a single dummy index, i. When written in index notation, a scalar-valued function will have no free indices, and a vector valued function will have a single free index. In the general case, an nth-order tensor-valued function will have n free indices. From the definition of the dot product, we can see that the dot product of two perpendicular vectors ($\theta_{\boldsymbol{ab}}=90°$) is equal to zero. In addition, the dot product of a vector with itself gives the magnitude of the vector squared, $|\boldsymbol{a}|^2=\boldsymbol{a}\cdot\boldsymbol{a}$. The dot product of a unit vector with itself will then be equal to one, $\boldsymbol{e}_1\cdot\boldsymbol{e}_1=1$. An ***orthonormal basis set*** has the property that each basis vector is perpendicular to the others. Therefore, the dot product of each basis vector with all other basis vectors is zero and the dot product of each basis vector with itself is equal to one giving

$$\boldsymbol{e}_i\cdot\boldsymbol{e}_j=\delta_{ij},$$

where we have introduced the ***Kronecker delta***, δ_{ij}, which has the property

$$\delta_{ij}=\begin{cases}0 & if \quad i\neq j\\ 1 & if \quad i=j\end{cases}. \tag{1.2}$$

The components of a vector, $\boldsymbol{a}$, along the direction of a unit vector $\boldsymbol{e}_1$, is given by

$$\boldsymbol{a}\cdot\boldsymbol{e_1}=|\boldsymbol{a}|\cos\theta_{a\boldsymbol{e}_1}=a_1,$$

where $\theta_{a\mathbf{e}_1}$ is the angle between vector $\mathbf{a}$ and the basis vector $\boldsymbol{e}_1$.

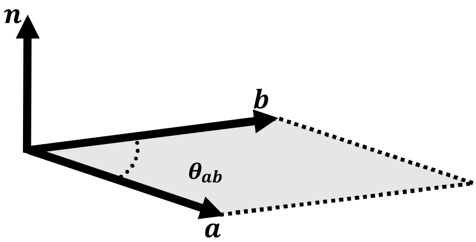

Figure 1.1. Illustration of a parallelogram bounded by vectors $\boldsymbol{a}$ and $\boldsymbol{b}$.

The result of the ***vector product*** or ***cross product***, $\boldsymbol{c}$, of two vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$, is a vector that is perpendicular to each of the original vectors. The cross product is written as

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b} = (a_2 b_3 - a_3 b_2)\,\boldsymbol{e}_1 + (a_3 b_1 - a_1 b_3)\,\boldsymbol{e}_2 + (a_1 b_2 - a_2 b_1)\,\boldsymbol{e}_3.$$

The magnitude of the cross product is equal to

$$|\boldsymbol{c}| = |\boldsymbol{a}||\boldsymbol{b}| \sin\theta_{ab},$$

where θ_{ab} is the angle between the two vectors.

The magnitude of the cross product is a measure of the area within a parallelogram defined by the two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, Figure 1.1. The unit normal perpendicular to the parallelogram is defined by the direction of the cross product, $\boldsymbol{n} = \frac{\boldsymbol{a}\times\boldsymbol{b}}{|\boldsymbol{a}\times\boldsymbol{b}|}$. Two parallel vectors will have a cross product equal to zero.

In this textbook, we will always employ a ***right-handed orthonormal basis set***, which has the properties that each basis vector is perpendicular to the other two, the magnitude of each basis vector is equal to one, and the basis vectors are related according to $\boldsymbol{e}_1 \times \boldsymbol{e}_2 = \boldsymbol{e}_3$. If these conditions are satisfied, the cross product between any two unit vectors can be written as

$$\boldsymbol{e}_i \times \boldsymbol{e}_j = \varepsilon_{ijk}\boldsymbol{e}_k,$$

where ε_{ijk} is the ***Levi-Civita symbol***, also known as the ***permutation symbol*** or the ***alternating symbol.*** The Levi-Civita symbol has the values

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if} \quad ijk = 123, 231, \text{ or } 312 \\ -1 & \text{if} \quad ijk = 132, 213, \text{ or } 321 \\ 0 & \quad\ \ \text{for repeated indices} \end{cases}.$$

A commonly used identity relating the permutation symbol and the Kronecker delta is

$$\varepsilon_{ijk}\varepsilon_{ipq} = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}. \tag{1.3}$$

EXAMPLE 1.1. *Determine whether each term in the following equation is a scalar, vector, or tensor and identify the free and dummy indices.*

$$B_{ij} = a_m a_m I_{ij} + \beta C_{ij}.$$

Solution:
The indices i and j both appear exactly once within each term of this equation. They are each free indices. The index m appears exactly twice within the second term. This is a dummy index and signifies a summation over the index m. The summation may be expanded to obtain

$$B_{ij} = (a_1 a_1 + a_2 a_2 + a_3 a_3)\, I_{ij} + \beta C_{ij}.$$

The variable, a_m, has a single index which signifies that a_m is the scalar component of the vector $\mathbf{a}$. The variable, B_{ij}, has two indices, which means B_{ij} is a scalar component of the second-order tensor, $\mathbf{B}$. The variable β has no index and is therefore a scalar.

EXAMPLE 1.2. *Find the value of* δ_{ii}.

Solution:
Expanding this equation using the summation convention, we find that

$$\begin{aligned} \delta_{ii} &= \sum_{i=1}^{3} \delta_{ii} \\ &= \delta_{11} + \delta_{22} + \delta_{33} \\ &= 3. \end{aligned}$$

EXAMPLE 1.3. *Show that* $\delta_{ij} a_i = a_j$.

Solution:
In this equation, there is both a dummy index, i and a free index, j. Therefore, this is a compact representation of the following three equations:

$$\begin{aligned} \delta_{i1} a_i &= \delta_{11} a_1 + \delta_{21} a_2 + \delta_{31} a_3 \\ &= 1 \times a_1 + 0 \times a_2 + 0 \times a_3 \\ &= a_1, \\ \delta_{i2} a_i &= \delta_{12} a_1 + \delta_{22} a_2 + \delta_{32} a_3 \\ &= 0 \times a_1 + 1 \times a_2 + 0 \times a_3 \\ &= a_2, \\ \delta_{i3} a_i &= \delta_{13} a_1 + \delta_{23} a_2 + \delta_{33} a_3 \\ &= 0 \times a_1 + 0 \times a_2 + 1 \times a_3 \\ &= a_3. \end{aligned}$$

This result can be compactly written as

$$\delta_{ij} a_i = a_j.$$

EXAMPLE 1.4. *Express the square of the magnitude of a vector in terms of its components.*

Solution:
We begin with the definition of the magnitude of a vector $|\boldsymbol{a}| = \sqrt{\boldsymbol{a} \cdot \boldsymbol{a}}$. Squaring this gives

$$|\boldsymbol{a}|^2 = \boldsymbol{a} \cdot \boldsymbol{a}.$$

The vector, $\boldsymbol{a}$, may be written in terms of its components as $\boldsymbol{a} = a_i \boldsymbol{e_i}$. Note that we cannot use the same index for the vector on the left and right of the dot product. Each is independently equal to the sum of the projections along each basis vector such that

$$|\boldsymbol{a}|^2 = a_i e_i \cdot a_j e_j.$$

The components, a_i and a_j, are scalar terms and commute freely. The dot product of the basis vectors e_i and e_j gives the Kronecker delta such that

$$\begin{aligned} |\boldsymbol{a}|^2 &= a_i a_j \delta_{ij} \\ &= a_i a_i \\ &= a_1^2 + a_2^2 + a_3^2. \end{aligned}$$

EXAMPLE 1.5. *Obtain an equation for the cross product of the two vectors* $\boldsymbol{a}$ *and* $\boldsymbol{b}$ *in terms of the components of the vectors.*

Solution:
The cross product, $\boldsymbol{c}$, of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ can be written as

$$\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}.$$

Expanding the vectors in terms of their components gives

$$\boldsymbol{c} = a_i \boldsymbol{e}_i \times b_j \boldsymbol{e}_j.$$

The components a_i and b_j are scalars and can be shifted in the equation to give

$$\boldsymbol{c} = a_i b_j \left(\boldsymbol{e}_i \times \boldsymbol{e}_j \right).$$

The cross product of the two basis vectors can be written in terms of the permutation symbol as

$$\varepsilon_{ijk} \boldsymbol{e}_k = \boldsymbol{e}_i \times \boldsymbol{e}_j,$$

which gives the result

$$\begin{aligned} \boldsymbol{c} &= a_i b_j \varepsilon_{ijk} \boldsymbol{e}_k \\ &= (a_2 b_3 - a_3 b_2)\boldsymbol{e}_1 + (a_3 b_1 - a_1 b_3)\boldsymbol{e}_2 + (a_1 b_2 - a_2 b_1)\boldsymbol{e}_3. \end{aligned}$$

EXAMPLE 1.6. *Show that* $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$.

Solution:
Drawing from the previous example, we can write

$$\boldsymbol{d} = \boldsymbol{b} \times \boldsymbol{c} = \varepsilon_{jmn} b_m c_n \boldsymbol{e}_j.$$

The components of the vector $\boldsymbol{d}$ are given by

$$d_j = \varepsilon_{jmn} b_m c_n.$$

We can also write

$$\boldsymbol{a} \times (\boldsymbol{d}) = \varepsilon_{ijk} a_i d_j \boldsymbol{e}_k.$$

Substituting the components of $\boldsymbol{d}$ into this equation gives the result

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \varepsilon_{ijk} a_i \left(\varepsilon_{mnj} b_m c_n\right) \boldsymbol{e}_k.$$

The scalar terms can be rearranged to give

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \varepsilon_{kij} \varepsilon_{mnj} a_i b_m c_n \boldsymbol{e}_k.$$

Using the identity in Equation (1.3), we obtain

$$\begin{aligned}\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) &= (\delta_{km}\delta_{in} - \delta_{kn}\delta_{im}) a_i b_m c_n \boldsymbol{e}_k \\ &= a_n b_k c_n \boldsymbol{e}_k - a_m b_m c_k \boldsymbol{e}_k \\ &= (a_n c_n) b_k \boldsymbol{e}_k - (a_m b_m) c_k \boldsymbol{e}_k \\ &= (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}.\end{aligned}$$

The final step makes use of the fact that $a_n c_n$ and $a_m b_m$ represent the dot product of two vectors, whereas $b_k \boldsymbol{e}_k$ and $c_k \boldsymbol{e}_k$ are component expansions of the vectors $\boldsymbol{b}$ and $\boldsymbol{c}$.

EXAMPLE 1.7. *Find the volume enclosed by the rhombus defined by vectors $\boldsymbol{a}, \boldsymbol{b}$, and $\boldsymbol{c}$.*

Solution:
The volume of a rhombus is simply the base area multiplied by the height perpendicular to the base:

$$Volume = height \times base\,area.$$

The area of the base can be found by taking the magnitude of the cross product of the bounding vectors, $base\,area = |\boldsymbol{a} \times \boldsymbol{b}|$. The height perpendicular to the base

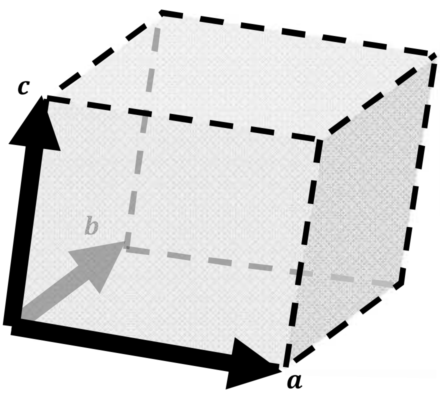

Figure 1.2. Illustration of a rhombus bounded by vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$.

is found by projecting the vector, $\boldsymbol{c}$, onto the vector perpendicular to the base, $\boldsymbol{n}$. Substitution gives

$$V = (\boldsymbol{c}\cdot\boldsymbol{n})(|\boldsymbol{a}\times\boldsymbol{b}|).$$

Since $|\boldsymbol{a}\times\boldsymbol{b}|$ is a scalar, we can slide it into the dot product

$$V = (\boldsymbol{c}\cdot|\boldsymbol{a}\times\boldsymbol{b}|\boldsymbol{n}).$$

Finally, we note that the normal to the base, $\boldsymbol{n}$, is equal to the magnitude of the cross product of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $\boldsymbol{n} = \frac{\boldsymbol{a}\times\boldsymbol{b}}{|\boldsymbol{a}\times\boldsymbol{b}|}$. Therefore, we can write

$$V = \boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b}).$$

1.2 Second-Order Tensors

A second-order tensor is used to represent a physical quantity whose representation requires a magnitude and two directions. For example, each component of the stress tensor is the resolution of a traction vector onto a surface. One needs the magnitude of the traction vector, as well as the direction of both the traction vector and the normal of the surface in order to define each component of the stress tensor. A more general and mathematically rigorous definition would be that a tensor, $\boldsymbol{A}$, is a linear operator that transforms a vector, $\boldsymbol{b}$, into another vector, $\boldsymbol{a}$,

$$\boldsymbol{a} = \boldsymbol{A}\cdot\boldsymbol{b}.$$

In the remainder of the textbook, we will use bolded capital roman letters to signify tensors while using lowercase bold letters to signify vectors. The only exception will be $\boldsymbol{X}$, which will be reserved for a vector. The ***tensor product*** or ***dyadic product*** of two vectors, $\boldsymbol{a}$ and $\boldsymbol{b}$, operating on any vector $\boldsymbol{c}$, produces a vector along the direction of $\boldsymbol{a}$ such that

$$(\boldsymbol{a}\otimes\boldsymbol{b})\cdot\boldsymbol{c} = (\boldsymbol{b}\cdot\boldsymbol{c})\,\boldsymbol{a}. \tag{1.4}$$

The components of the dyadic product of two vectors are given by

$$\left(\boldsymbol{a}\otimes\boldsymbol{b}\right)_{ij} = a_i b_j.$$

Any second-order tensor may be expressed as a linear combination of the dyadic product of the basis vectors as

$$\boldsymbol{A} = A_{ij}\boldsymbol{e}_i\otimes\boldsymbol{e}_j\,, \tag{1.5}$$

where A_{ij} are the scalar components of the tensor, $\boldsymbol{A}$, within the basis set $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$. These components may be expressed as a 3×3 matrix

$$[\boldsymbol{A}] = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}.$$

The component indices, A_{ij}, represent the matrix row, i, and the matrix column, j. Whereas we have expressed the components of a tensor as a matrix, not all 3×3 matrices are tensors. The physical quantity, such as stress or velocity, measured by either a vector or a tensor remains invariant when the coordinate system is changed.

The vector representing velocity does not change, only the components expressed within the coordinate system change. The same must be true for any higher order tensor. In fact, the transformation relating components of any tensor within two reference frames is quite exact and will be explored in more detail in a later section.

The majority of the equations within this textbook will require the manipulation of tensor and vector equations. A number of useful tensor and vector identities can be found in Table 12.3. In this section, we will introduce the commonly used tensor notation. The ***transpose***, $\boldsymbol{A}^T$, of a tensor, $\boldsymbol{A}$, is denoted by the superscript T and is defined in terms of the components of the original tensor $\boldsymbol{A}$ by

$$\boldsymbol{A}^T = A_{\boldsymbol{ji}}\boldsymbol{e_i}\otimes\boldsymbol{e}_j = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}.$$

If a tensor is defined as a dyadic product of two vectors, $\boldsymbol{A} = \boldsymbol{a}\otimes\boldsymbol{b}$, then the transpose is given by $\boldsymbol{A}^T = \boldsymbol{b}\otimes\boldsymbol{a}$. Generally, a tensor is not equal to its transpose and the dyadic product does not commute

$$\boldsymbol{a}\otimes\boldsymbol{b} \neq \boldsymbol{b}\otimes\boldsymbol{a}.$$

There are special names reserved for tensors, which possess additional symmetry properties. A ***symmetric tensor*** is a tensor that is equal to its transpose giving $\boldsymbol{A} = \boldsymbol{A}^T$. A ***skew-symmetric tensor*** is a tensor that is equal to the negative of its transpose giving $\boldsymbol{A} = -\boldsymbol{A}^T$. Notice that the condition for a skew-symmetric tensor requires that each diagonal term of the tensor is equal to its negative. This requires that each diagonal term must be equal to zero for the skew-symmetric tensor. In later sections, we will introduce a number of physical quantities that may be represented by symmetric tensors such as the Cauchy stress and infinitesimal strain.

It is sometimes convenient to take a general tensor, $\boldsymbol{B}$, which has no symmetry properties, and break it up into a symmetric, $\boldsymbol{B}^s$, and a skew-symmetric tensor, $\boldsymbol{B}^a$. The superscript a stands for antisymmetric. This can be done through a simple decomposition of the tensor, $\boldsymbol{B}$,

$$\boldsymbol{B} = \frac{1}{2}(\boldsymbol{B}+\boldsymbol{B}^T) + \frac{1}{2}(\boldsymbol{B}-\boldsymbol{B}^T).$$

The first bracketed term gives the symmetric part, $\boldsymbol{B}^s = \frac{1}{2}(\boldsymbol{B}+\boldsymbol{B}^T)$ of the tensor, whereas the second bracketed term gives the skew-symmetric part, $\boldsymbol{B}^a = \frac{1}{2}(\boldsymbol{B}-\boldsymbol{B}^T)$.

The ***trace*** of a tensor is the sum of the diagonal components of the tensor

$$tr\,(\boldsymbol{A}) = A_{11} + A_{22} + A_{33} = A_{ii}.$$

We also introduce a new operator, :, which represents the ***contraction*** of two tensors

$$\boldsymbol{A}:\boldsymbol{B} = tr(\mathbf{A}^T\cdot\mathbf{B}) = A_{ij}B_{ij}.$$

The contraction of two tensors gives a scalar. The contraction of any symmetric tensor with any skew-symmetric tensor is always equal to zero. In the case of the decomposition described earlier, we have $\boldsymbol{B}^s:\boldsymbol{B}^a = 0$. We will use the contraction operator to define the ***tensor norm***, $||\boldsymbol{A}||$, as

$$||\boldsymbol{A}|| = \sqrt{(\boldsymbol{A}:\boldsymbol{A})}.$$

This norm has the property that it is a positive scalar and has a value of zero if and only if each of the components of the tensor $\boldsymbol{A}$ is zero.

The ***determinant*** of a tensor is given by

$$\det(\boldsymbol{A}) = \epsilon_{ijk} A_{i1} A_{j2} A_{k3}.$$

The determinate of a tensor is equal to the determinate of its transpose

$$\det(\boldsymbol{A}) = \det(\boldsymbol{A}^T)$$

A tensor is ***singular*** if and only if the determinant of that tensor is zero. Any ***nonsingular*** tensor will possess a unique ***inverse***, $\boldsymbol{A}^{-1}$, which satisfies

$$\boldsymbol{A} \cdot \boldsymbol{A}^{-\mathbf{1}} = \mathbf{I}.$$

As mentioned earlier, a tensor is a linear operator that gives a vector when acting on a vector. Assuming we have a nonsingular tensor, it uniquely transforms vectors from one vector space to another vector space. An ***orthogonal tensor***, $\boldsymbol{Q}$, acting on a vector will change the direction but not the magnitude of the vector

$$|\boldsymbol{x}| = |\boldsymbol{Q} \cdot \boldsymbol{x}|.$$

This is a tensor that rotates vectors from one vector space into another. Orthogonal tensors have the property that the transpose is equal to the inverse

$$\boldsymbol{Q}^{\boldsymbol{T}} \cdot \boldsymbol{Q} = \mathbf{I}. \tag{1.6}$$

This has the interesting implication that the determinant of the orthogonal tensor must be equal to plus or minus one. This is obtained by taking the determinant of Equation (1.6). This gives

$$\det(\boldsymbol{Q}^T \cdot \boldsymbol{Q}) = \det(\mathbf{I})$$
$$\det(\boldsymbol{Q}^T)\det(\boldsymbol{Q}) = 1$$
$$(\det(\boldsymbol{Q}))^{\mathbf{2}} = 1.$$

If the determinant of the tensor is equal to one, the tensor is a ***proper orthogonal tensor***. A proper orthogonal tensor preserves the right-handedness (or left-handedness) of a coordinate system. An ***improper orthogonal tensor*** acting on a system will invert the spatial relationship between vectors.

EXAMPLE 1.8. *Given the tensor, $\boldsymbol{A}$, where*

$$\boldsymbol{A} = \begin{bmatrix} 4 & 3 & 9 \\ -5 & 2 & 3 \\ 7 & 6 & -4 \end{bmatrix},$$

find a) the transpose, $\boldsymbol{A}^T$, b) the trace, tr$\boldsymbol{A}$, c) the tensor norm, $\|\mathbf{A}\|$, and d) the symmetric and skew-symmetric tensors $\mathbf{A}^s$ and $\mathbf{A}^a$ such that $\mathbf{A} = \mathbf{A}^s + \mathbf{A}^a$.

Solution:

a) $\boldsymbol{A}^T = \begin{bmatrix} 4 & -5 & 7 \\ 3 & 2 & 6 \\ 9 & 3 & -4 \end{bmatrix}$