Computer Vision
Models, Learning, and Inference

This modern treatment of computer vision focuses on learning and inference in probabilistic models as a unifying theme. It shows how to use training data to learn the relationships between the observed image data and the aspects of the world that we wish to estimate, such as the 3D structure or the object class, and how to exploit these relationships to make inferences about the world from new image data.

With minimal prerequisites, the book starts from the basics of probability and model fitting and works up to real examples that the reader can implement and modify to build useful vision systems. Primarily meant for advanced undergraduate and graduate students, the detailed methodological presentation will also be useful for practitioners of computer vision.

– Covers cutting-edge techniques, including graph cuts, machine learning, and multiple view geometry.
– A unified approach shows the common basis for solutions of important computer vision problems, such as camera calibration, face recognition, and object tracking.
– More than 70 algorithms are described in sufficient detail to implement.
– More than 350 full-color illustrations amplify the text.
– The treatment is self-contained, including all of the background mathematics.
– Additional resources at www.computervisionmodels.com.

Dr. Simon J. D. Prince is a faculty member in the Department of Computer Science at University College London. He has taught courses on machine vision, image processing, and advanced mathematical methods. He has a diverse background in biological and computing sciences and has published papers across the fields of computer vision, biometrics, psychology, physiology, medical imaging, computer graphics, and HCI.
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Models, Learning, and Inference

Simon J. D. Prince
University College London
This book is dedicated to Richard Eagle, without whom it would never have been started, and to Lynfa Stroud, without whom it would never have been finished.
# Contents

Acknowledgments  
Foreword by Andrew Fitzgibbon  
Preface  

## 1 Introduction

1. Organization of the book  
2. Other books  

## 2 Introduction to probability

1. Random variables  
2. Joint probability  
3. Marginalization  
4. Conditional probability  
5. Bayes’ rule  
6. Independence  
7. Expectation  

## 3 Common probability distributions

1. Bernoulli distribution  
2. Beta distribution  
3. Categorical distribution  
4. Dirichlet distribution  
5. Univariate normal distribution  
6. Normal-scaled inverse gamma distribution  
7. Multivariate normal distribution  
8. Normal inverse Wishart distribution  
9. Conjugacy  

## 4 Fitting probability models

1. Maximum likelihood  
2. Maximum a posteriori  
3. The Bayesian approach  
4. Worked example 1: Univariate normal  
5. Worked example 2: Categorical distribution
<table>
<thead>
<tr>
<th>Contents</th>
</tr>
</thead>
<tbody>
<tr>
<td>5 The normal distribution</td>
</tr>
<tr>
<td>5.1 Types of covariance matrix</td>
</tr>
<tr>
<td>5.2 Decomposition of covariance</td>
</tr>
<tr>
<td>5.3 Linear transformations of variables</td>
</tr>
<tr>
<td>5.4 Marginal distributions</td>
</tr>
<tr>
<td>5.5 Conditional distributions</td>
</tr>
<tr>
<td>5.6 Product of two normals</td>
</tr>
<tr>
<td>5.7 Change of variable</td>
</tr>
<tr>
<td>II Machine learning for machine vision</td>
</tr>
<tr>
<td>6 Learning and inference in vision</td>
</tr>
<tr>
<td>6.1 Computer vision problems</td>
</tr>
<tr>
<td>6.2 Types of model</td>
</tr>
<tr>
<td>6.3 Example 1: Regression</td>
</tr>
<tr>
<td>6.4 Example 2: Binary classification</td>
</tr>
<tr>
<td>6.5 Which type of model should we use?</td>
</tr>
<tr>
<td>6.6 Applications</td>
</tr>
<tr>
<td>7 Modeling complex data densities</td>
</tr>
<tr>
<td>7.1 Normal classification model</td>
</tr>
<tr>
<td>7.2 Hidden variables</td>
</tr>
<tr>
<td>7.3 Expectation maximization</td>
</tr>
<tr>
<td>7.4 Mixture of Gaussians</td>
</tr>
<tr>
<td>7.5 The t-distribution</td>
</tr>
<tr>
<td>7.6 Factor analysis</td>
</tr>
<tr>
<td>7.7 Combining models</td>
</tr>
<tr>
<td>7.8 Expectation maximization in detail</td>
</tr>
<tr>
<td>7.9 Applications</td>
</tr>
<tr>
<td>8 Regression models</td>
</tr>
<tr>
<td>8.1 Linear regression</td>
</tr>
<tr>
<td>8.2 Bayesian linear regression</td>
</tr>
<tr>
<td>8.3 Nonlinear regression</td>
</tr>
<tr>
<td>8.4 Kernels and the kernel trick</td>
</tr>
<tr>
<td>8.5 Gaussian process regression</td>
</tr>
<tr>
<td>8.6 Sparse linear regression</td>
</tr>
<tr>
<td>8.7 Dual linear regression</td>
</tr>
<tr>
<td>8.8 Relevance vector regression</td>
</tr>
<tr>
<td>8.9 Regression to multivariate data</td>
</tr>
<tr>
<td>8.10 Applications</td>
</tr>
<tr>
<td>9 Classification models</td>
</tr>
<tr>
<td>9.1 Logistic regression</td>
</tr>
<tr>
<td>9.2 Bayesian logistic regression</td>
</tr>
<tr>
<td>9.3 Nonlinear logistic regression</td>
</tr>
<tr>
<td>9.4 Dual logistic regression</td>
</tr>
<tr>
<td>9.5 Kernel logistic regression</td>
</tr>
</tbody>
</table>
## Contents

9.6 Relevance vector classification .................................. 147
9.7 Incremental fitting and boosting .................................. 150
9.8 Classification trees .................................................. 153
9.9 Multiclass logistic regression ...................................... 156
9.10 Random trees, forests, and ferns ................................. 158
9.11 Relation to non-probabilistic models ............................ 159
9.12 Applications .......................................................... 160

### III Connecting local models

10 Graphical models ................................................... 173
  10.1 Conditional independence ....................................... 173
  10.2 Directed graphical models ...................................... 175
  10.3 Undirected graphical models ................................... 178
  10.4 Comparing directed and undirected graphical models ....... 181
  10.5 Graphical models in computer vision .......................... 181
  10.6 Inference in models with many unknowns ..................... 184
  10.7 Drawing samples .................................................. 186
  10.8 Learning ............................................................ 188

11 Models for chains and trees ........................................ 195
  11.1 Models for chains ................................................ 196
  11.2 MAP inference for chains ...................................... 198
  11.3 MAP inference for trees ......................................... 202
  11.4 Marginal posterior inference for chains ....................... 205
  11.5 Marginal posterior inference for trees ......................... 211
  11.6 Learning in chains and trees .................................. 212
  11.7 Beyond chains and trees ........................................ 213
  11.8 Applications ........................................................ 216

12 Models for grids ..................................................... 227
  12.1 Markov random fields ............................................ 228
  12.2 MAP inference for binary pairwise MRFs ...................... 231
  12.3 MAP inference for multilabel pairwise MRFs .................. 239
  12.4 Multilabel MRFs with non-convex potentials .................. 244
  12.5 Conditional random fields ...................................... 247
  12.6 Higher order models ............................................. 250
  12.7 Directed models for grids ...................................... 250
  12.8 Applications ........................................................ 251

### IV Preprocessing

13 Image preprocessing and feature extraction ........................ 269
  13.1 Per-pixel transformations ...................................... 269
  13.2 Edges, corners, and interest points ........................... 279
  13.3 Descriptors ....................................................... 283
  13.4 Dimensionality reduction ....................................... 287
### V Models for geometry

#### 14 The pinhole camera
- 14.1 The pinhole camera ........................................ 297
- 14.2 Three geometric problems ..................................... 304
- 14.3 Homogeneous coordinates ..................................... 306
- 14.4 Learning extrinsic parameters ................................. 309
- 14.5 Learning intrinsic parameters ................................. 311
- 14.6 Inferring three-dimensional world points ..................... 312
- 14.7 Applications .................................................. 314

#### 15 Models for transformations
- 15.1 Two-dimensional transformation models ..................... 323
- 15.2 Learning in transformation models ............................ 330
- 15.3 Inference in transformation models ............................ 334
- 15.4 Three geometric problems for planes ......................... 335
- 15.5 Transformations between images ............................... 339
- 15.6 Robust learning of transformations ............................ 342
- 15.7 Applications .................................................. 347

#### 16 Multiple cameras
- 16.1 Two-view geometry ........................................... 355
- 16.2 The essential matrix .......................................... 357
- 16.3 The fundamental matrix ...................................... 361
- 16.4 Two-view reconstruction pipeline ............................. 364
- 16.5 Rectification .................................................. 368
- 16.6 Multiview reconstruction ..................................... 372
- 16.7 Applications .................................................. 376

### VI Models for vision

#### 17 Models for shape
- 17.1 Shape and its representation .................................. 388
- 17.2 Snakes ....................................................... 389
- 17.3 Shape templates .............................................. 393
- 17.4 Statistical shape models ..................................... 396
- 17.5 Subspace shape models ...................................... 399
- 17.6 Three-dimensional shape models ............................. 405
- 17.7 Statistical models for shape and appearance ................. 405
- 17.8 Non-Gaussian statistical shape models ....................... 410
- 17.9 Articulated models .......................................... 414
- 17.10 Applications ................................................ 415

#### 18 Models for style and identity
- 18.1 Subspace identity model ..................................... 427
- 18.2 Probabilistic linear discriminant analysis .................... 433
- 18.3 Nonlinear identity models .................................... 437
- 18.4 Asymmetric bilinear models .................................. 438
## Contents

18.5 Symmetric bilinear and multilinear models .......... 443  
18.6 Applications ..................................... 446  

19 Temporal models ..................................... 453  
19.1 Temporal estimation framework ................. 453  
19.2 Kalman filter .................................... 455  
19.3 Extended Kalman filter ......................... 466  
19.4 Unscented Kalman filter ....................... 467  
19.5 Particle filtering .................................. 472  
19.6 Applications ..................................... 476  

20 Models for visual words ............................. 483  
20.1 Images as collections of visual words .......... 483  
20.2 Bag of words .................................... 484  
20.3 Latent Dirichlet allocation ..................... 487  
20.4 Single author–topic model ...................... 493  
20.5 Constellation models ............................. 495  
20.6 Scene models .................................... 499  
20.7 Applications ..................................... 500  

VII Appendices .......................................... 507  

A Notation .............................................. 507  

B Optimization .......................................... 509  
B.1 Problem statement .................................. 509  
B.2 Choosing a search direction ...................... 511  
B.3 Line search ........................................ 515  
B.4 Reparameterization .................................. 516  

C Linear algebra ......................................... 519  
C.1 Vectors .............................................. 519  
C.2 Matrices .............................................. 520  
C.3 Tensors .............................................. 522  
C.4 Linear transformations ............................. 522  
C.5 Singular value decomposition ................... 522  
C.6 Matrix calculus ..................................... 527  
C.7 Common problems .................................. 528  
C.8 Tricks for inverting large matrices .......... 530  

Bibliography ............................................. 533  

Index ...................................................... 567
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Foreword

I was very pleased to be asked to write this foreword, having seen snapshots of the development of this book since its inception. I write this having just returned from BMVC 2011, where I found that others had seen draft copies, and where I heard comments like “What amazing figures!”, “It’s so comprehensive!”, and “He’s so Bayesian!”.

But I don’t want you to read this book just because it has amazing figures and provides new insights into vision algorithms of every kind, or even because it’s “Bayesian” (although more on that later). I want you to read it because it makes clear the most important distinction in computer vision research: the difference between “model” and “algorithm.” This is akin to the distinction that Marr made with his three-level computational theory, but Prince’s two-level distinction is made beautifully clear by his use of the language of probability.

Why is this distinction so important? Well, let us look at one of the oldest and apparently easiest problems in vision: separating an image into “figure” and “ground.” It is still common to hear students new to vision address this problem just as the early vision researchers did, by reciting an algorithm: first I’ll use PCA to find the dominant color axis, then I’ll generate a grayscale image, then I’ll threshold that at some value, then I’ll clean up the holes using morphological operators. Trying their recipe on some test images, the novice discovers that real images are rather more complicated, so new steps are added: I’ll need some sort of adaptive threshold, I can get that by blurring the edge map and locally computing maxima.

However, as most readers will already know, such recipes are extremely brittle, meaning that the various “magic numbers” controlling each step all interact, making it impossible to find a set of parameters that works for all images (or even a useful subset). The root of this problem is that the objective of the algorithm has never been defined. What do we mean by figure and ground separation? Can we specify what we mean mathematically?

When vision researchers began to address these problems, the language of statistics and Markov random fields allowed a clean distinction between the objective and the algorithm to be drawn. We write down not the steps to solve the problem but the problem itself, for example, as a function to be minimized. In the language of this book, we write down formulae for all the probability distributions that define the problem and then perform operations on those distributions in order to provide answers. This book shows how this can be done for a huge variety of vision problems and how doing so provides more robust solutions that are much easier to reason about.

This is not to say that one can just write down the model and ask others to solve for its parameters because the space of possible models is so much vaster than the space of ones in which the solution is tractable. Thus, one always has at the back of one’s mind a collection of models known to be soluble, and one always tries to find a model for one’s problem, which is near some soluble problem. At that stage, one may well think in
terms of strategies such as “I can probably generalize alpha expansion a bit to solve for the discrete parameters, and then I can use a Gauss-Newton method for the continuous ones, and that will probably be slow, but it will tell me if it’s worth trying to invent a faster combined algorithm.” Such strategies are common and can be helpful, provided one always retains an idea of the model underlying them.

However, even armed with the attitudes this book will engender, experienced researchers today can fall into the trap of failing to distinguish model and algorithm. They find themselves thinking thoughts like: “I’ll fit a mixture of Gaussians to the color distribution. Then I’ll model the mixture weights as an MRF and use graph cuts to update them. Then I’ll go back to step 1 and repeat.” The good news is that often such recipes can be turned back into models. Even if the only known way of fitting the model is to use the recipe you just thought of, the discipline of thinking of it as a model allows you to reason about it, to make use of alternative techniques, and ultimately to do better research. Reading this book is a sure way to improve your ability to make that jump.

So what is this language of probabilities that will allow us to become better researchers? Well, let me provide my “Engineer’s view of Bayes’ theorem.” It is common to hear a distinction between “Bayesians” and “Frequentists,” but I think many engineers have a much more fundamental problem with Bayes: Bayesians must lie. Their estimates, biased toward the prior mean, are deliberately different from the most probable reading of their sensor. Consider the example of an “I speak your height” machine whose sensor has a uniformly distributed ±1 cm error. You receive £1 every time you correctly predict someone’s height to within 1 cm. You receive £1 every time you correctly predict someone’s height to within 1 cm. Bayesian principles suggest that if your sensor reads 200 cm ± 1 cm, you should report 199 cm; you will make more money than guessing the actual sensor reading, because more 199 cm people will appear than those of 200 cm. So I as an engineer believe in Bayes as a way of getting better answers, and thus very much welcome this book’s pragmatic (but much more subtle than mine) embrace of Bayes. I wonder if it might even be considered a book on statistics with vision examples rather than a book on vision built on probability.

But it would be wrong to finish this foreword without mentioning the figures. They really are good, not because they’re beautiful (they often are), but because they provide crucial insights into the workings of even the most basic of algorithms and ideas. The illustrations in Chapters 2–4 are fundamental to the understanding of modern Bayesian inference, and yet I doubt that there are more than a handful of researchers who have ever seen them all. Later figures express extremely complex ideas more clearly than I have ever seen, as well as representing fabulously “clean” implementations of fundamental algorithms, which really show us how the underlying models influence our capabilities.

Finally I believe it is worth directly comparing this book to the recent textbook by my colleague Richard Szeliski. That book too is marked by an enormously comprehensive view of computer vision, by excellent illustration, by insightful notation, and intellectual synthesis of large groups of existing ideas. But in a real sense the two books operate at opposite ends of the pedagogical spectrum: Szeliski is a comprehensive summary of the state of the art in computer vision, the frontier of our knowledge and abilities, while this book addresses the fundamentals of how we make progress in this challenging and exciting field. I look forward to many decades with both on my shelf, or indeed, I suspect, open on my desktop.

Andrew Fitzgibbon
Microsoft Research, Cambridge
September 2011
Preface

There are already many computer vision textbooks, and it is reasonable to question the need for another. Let me explain why I chose to write this volume.

Computer vision is an engineering discipline; we are primarily motivated by the real-world concern of building machines that see. Consequently, we tend to categorize our knowledge by the real-world problem that it addresses. For example, most existing vision textbooks contain chapters on object recognition and stereo vision. The sessions at our research conferences are organized in the same way. The role of this book is to question this orthodoxy: Is this really the way that we should organize our knowledge?

Consider the topic of object recognition. A wide variety of methods have been applied to this problem (e.g., subspace models, boosting methods, bag of words models, and constellation models). However, these approaches have little in common. Any attempt to describe the grand sweep of our knowledge devolves into an unstructured list of techniques. How can we make sense of it all for a new student? I will argue for a different way to organize our knowledge, but first let me tell you how I see computer vision problems.

We observe an image and from this we extract measurements. For example, we might use the RGB values directly or we might filter the image or perform some more sophisticated preprocessing. The vision problem or goal is to use the measurements to infer the world state. For example, in stereo vision we try to infer the depth of the scene. In object detection, we attempt to infer the presence or absence of a particular class of object.

To accomplish the goal, we build a model. The model describes a family of statistical relationships between the measurements and the world state. The particular member of that family is determined by a set of parameters. In learning we choose these parameters so they accurately reflect the relationship between the measurements and the world. In inference we take a new set of measurements and use the model to tell us about the world state. The methods for learning and inference are embodied in algorithms. I believe that computer vision should be understood in these terms: the goal, the measurements, the world state, the model, the parameters, and the learning and inference algorithms.

We could choose to organize our knowledge according to any of these quantities, but in my opinion what is most critical is the model itself – the statistical relationship between the world and the measurements. There are three reasons for this. First, the model type often transcends the application (the same model can be used for diverse vision tasks). Second, the models naturally organize themselves neatly into distinct families (e.g., regression, Markov random fields, camera models) that can be understood in relative isolation. Finally, discussing vision on the level of models allows us to draw
connections between algorithms and applications that initially appear unrelated. Accordingly, this book is organized so that each main chapter considers a different family of models.

On a final note, I should say that I found most of the ideas in this book very hard to grasp when I was first exposed to them. My goal was to make this process easier for subsequent students following the same path; I hope that this book achieves this and inspires the reader to learn more about computer vision.