Random Matrix Methods for Wireless Communications

Blending theoretical results with practical applications, this book provides an introduction to random matrix theory and shows how it can be used to tackle a variety of problems in wireless communications. The Stieltjes transform method, free probability theory, combinatoric approaches, deterministic equivalents, and spectral analysis methods for statistical inference are all covered from a unique engineering perspective. Detailed mathematical derivations are presented throughout, with thorough explanations of the key results and all fundamental lemmas required for the readers to derive similar calculus on their own. These core theoretical concepts are then applied to a wide range of real-world problems in signal processing and wireless communications, including performance analysis of CDMA, MIMO, and multi-cell networks, as well as signal detection and estimation in cognitive radio networks. The rigorous yet intuitive style helps demonstrate to students and researchers alike how to choose the correct approach for obtaining mathematically accurate results.

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Random Matrix Methods for Wireless Communications

Romain Couillet and Mérouane Debbah
École Supérieure d'Electricité, Gif sur Yvette, France
To my family,
Romain Couillet

To my parents,
Merouane Debbah
Contents

Prefacepage xiii
Acknowledgments xv
Acronyms xvi
Notation xviii

1 Introduction 1
1.1 Motivation 1
1.2 History and book outline 6

Part I Theoretical aspects 15

2 Random matrices 17
2.1 Small dimensional random matrices 17
2.1.1 Definitions and notations 17
2.1.2 Wishart matrices 19
2.2 Large dimensional random matrices 29
2.2.1 Why go to infinity? 29
2.2.2 Limit spectral distributions 30

3 The Stieltjes transform method 35
3.1 Definitions and overview 35
3.2 The Marcenko Pastur law 42
3.2.1 Proof of the Marcenko Pastur law 44
3.2.2 Truncation, centralization, and rescaling 54
3.3 Stieltjes transform for advanced models 57
3.4 Tonelli theorem 61
3.5 Central limit theorems 63

4 Free probability theory 71
4.1 Introduction to free probability theory 72
4.2 $R$- and $S$-transforms 75
4.3 Free probability and random matrices 77
4.4 Free probability for Gaussian matrices 84
## Contents

4.5  Free probability for Haar matrices 87

5  Combinatoric approaches 95
5.1  The method of moments 95
5.2  Free moments and cumulants 98
5.3  Generalization to more structured matrices 105
5.4  Free moments in small dimensional matrices 108
5.5  Rectangular free probability 109
5.6  Methodology 111

6  Deterministic equivalents 113
6.1  Introduction to deterministic equivalents 113
6.2  Techniques for deterministic equivalents 115
  6.2.1  Bai and Silverstein method 115
  6.2.2  Gaussian method 139
  6.2.3  Information plus noise models 145
  6.2.4  Models involving Haar matrices 153
6.3  A central limit theorem 175

7  Spectrum analysis 179
7.1  Sample covariance matrix 180
  7.1.1  No eigenvalues outside the support 180
  7.1.2  Exact spectrum separation 183
  7.1.3  Asymptotic spectrum analysis 186
7.2  Information plus noise model 192
  7.2.1  Exact separation 192
  7.2.2  Asymptotic spectrum analysis 195

8  Eigen-inference 199
8.1  G-estimation 199
  8.1.1  Girko G-estimators 199
  8.1.2  G-estimation of population eigenvalues and eigenvectors 201
  8.1.3  Central limit for G-estimators 213
8.2  Moment deconvolution approach 218

9  Extreme eigenvalues 223
9.1  Spiked models 223
  9.1.1  Perturbed sample covariance matrix 224
  9.1.2  Perturbed random matrices with invariance properties 228
9.2  Distribution of extreme eigenvalues 230
  9.2.1  Introduction to the method of orthogonal polynomials 230
  9.2.2  Limiting laws of the extreme eigenvalues 233
9.3  Random matrix theory and eigenvectors 237
10 Summary and partial conclusions 243

Part II Applications to wireless communications 249

11 Introduction to applications in telecommunications 251
  11.1 Historical account of major results 251
      11.1.1 Rate performance of multi-dimensional systems 252
      11.1.2 Detection and estimation in large dimensional systems 256
      11.1.3 Random matrices and flexible radio 259

12 System performance of CDMA technologies 263
  12.1 Introduction 263
  12.2 Performance of random CDMA technologies 264
      12.2.1 Random CDMA in uplink frequency flat channels 264
      12.2.2 Random CDMA in uplink frequency selective channels 273
      12.2.3 Random CDMA in downlink frequency selective channels 281
  12.3 Performance of orthogonal CDMA technologies 284
      12.3.1 Orthogonal CDMA in uplink frequency flat channels 285
      12.3.2 Orthogonal CDMA in uplink frequency selective channels 285
      12.3.3 Orthogonal CDMA in downlink frequency selective channels 286

13 Performance of multiple antenna systems 293
  13.1 Quasi-static MIMO fading channels 293
  13.2 Time-varying Rayleigh channels 295
      13.2.1 Small dimensional analysis 296
      13.2.2 Large dimensional analysis 297
      13.2.3 Outage capacity 298
  13.3 Correlated frequency flat fading channels 300
      13.3.1 Communication in strongly correlated channels 305
      13.3.2 Ergodic capacity in strongly correlated channels 309
      13.3.3 Ergodic capacity in weakly correlated channels 311
      13.3.4 Capacity maximizing precoder 312
  13.4 Rician fading channels 316
      13.4.1 Quasi-static mutual information and ergodic capacity 316
      13.4.2 Capacity maximizing power allocation 318
      13.4.3 Outage mutual information 320
  13.5 Frequency selective channels 322
      13.5.1 Ergodic capacity 324
      13.5.2 Capacity maximizing power allocation 325
  13.6 Transceiver design 328
      13.6.1 Channel matrix model with i.i.d. entries 331
      13.6.2 Channel matrix model with generalized variance profile 332
## 14 Rate performance in multiple access and broadcast channels

14.1 Broadcast channels with linear precoders
   14.1.1 System model
   14.1.2 Deterministic equivalent of the SINR
   14.1.3 Optimal regularized zero-forcing precoding
   14.1.4 Zero-forcing precoding
   14.1.5 Applications

14.2 Rate region of MIMO multiple access channels
   14.2.1 MAC rate region in quasi-static channels
   14.2.2 Ergodic MAC rate region
   14.2.3 Multi-user uplink sum rate capacity

## 15 Performance of multi-cellular and relay networks

15.1 Performance of multi-cell networks
   15.1.1 Two-cell network
   15.1.2 Wyner model

15.2 Multi-hop communications
   15.2.1 Multi-hop model
   15.2.2 Mutual information
   15.2.3 Large dimensional analysis
   15.2.4 Optimal transmission strategy

## 16 Detection

16.1 Cognitive radios and sensor networks

16.2 System model

16.3 Neyman Pearson criterion
   16.3.1 Known signal and noise variances
   16.3.2 Unknown signal and noise variances
   16.3.3 Unknown number of sources

16.4 Alternative signal sensing approaches
   16.4.1 Condition number method
   16.4.2 Generalized likelihood ratio test
   16.4.3 Test power and error exponents

## 17 Estimation

17.1 Directions of arrival
   17.1.1 System model
   17.1.2 The MUSIC approach
   17.1.3 Large dimensional eigen-inference
   17.1.4 The correlated signal case

17.2 Blind multi-source localization
   17.2.1 System model
   17.2.2 Small dimensional inference
## Contents

17.2.3 Conventional large dimensional approach 438  
17.2.4 Free deconvolution approach 440  
17.2.5 Analytic method 447  
17.2.6 Joint estimation of number of users, antennas and powers 469  
17.2.7 Performance analysis 471  

18 System modeling 477  
18.1 Introduction to Bayesian channel modeling 478  
18.2 Channel modeling under environmental uncertainty 480  
18.2.1 Channel energy constraints 481  
18.2.2 Spatial correlation models 484  

19 Perspectives 501  
19.1 From asymptotic results to finite dimensional studies 501  
19.2 The replica method 505  
19.3 Towards time-varying random matrices 506  

20 Conclusion 511  
References 515  
Index 537
Preface

More than sixty years have passed since the 1948 landmark paper of Shannon providing the capacity of a single antenna point-to-point communication channel. The method was based on information theory and led to a revolution in the field, especially on how communication systems were designed. The tools then showed their limits when we wanted to extend the analysis and design to the multi-terminal multiple antenna case, which is the basis of the wireless revolution since the nineties. Indeed, in the design of these networks, engineers frequently stumble on the scalability problem. In other words, as the number of nodes or bandwidth increase, problems become harder to solve and the determination of the precise achievable rate region becomes an intractable problem. Moreover, engineering insight progressively disappears and we can only rely on heavy simulations with all their caveats and limitations. However, when the system is sufficiently large, we may hope that a macroscopic view could provide a more useful abstraction of the network. The properties of the new macroscopic model nonetheless need to account for microscopic considerations, e.g. fading, mobility, etc. We may then sacrifice some structural details of the microscopic view but the macroscopic view will preserve sufficient information to allow for a meaningful network optimization solution and the derivation of insightful results in a wide range of settings.

Recently, a number of research groups around the world have taken this approach and have shown how tools borrowed from physical and mathematical frameworks, e.g. percolation theory, continuum models, game theory, electrostatics, mean field theory, stochastic geometry, just to name a few, can capture most of the complexity of dense random networks in order to unveil some relevant features on network-wide behavior.

The following book falls within this trend and aims to provide a comprehensive understanding on how random matrix theory can model the complexity of the interaction between wireless devices. It has been more than fifteen years since random matrix theory was successfully introduced into the field of wireless communications to analyze CDMA and MIMO systems. One of the useful features, especially of the large dimensional random matrix theory approach, is its ability to predict, under certain conditions, the behavior of the empirical eigenvalue distribution of products and sums of matrices. The results are striking in terms of accuracy compared to simulations with reasonable matrix sizes, and
the theory has been shown to be an efficient tool to predict the behavior of wireless systems with only few meaningful parameters. Random matrix theory is also increasingly making its way into the statistical signal processing field with the generalization of detection and inference methods, e.g. array processing, hypothesis tests, parameter estimation, etc., to the multi-variate case. This comes as a small revolution in modern signal processing as legacy estimators, such as the MUSIC method, become increasingly obsolete and unadapted to large sensing arrays with few observations.

The authors are confident and have no doubt on the usefulness of the tool for the engineering community in the upcoming years, especially as networks become denser. They also think that random matrix theory should become sooner or later a major tool for electrical engineers, taught at the graduate level in universities. Indeed, engineering education programs of the twentieth century were mostly focused on the Fourier transform theory due to the omnipresence of frequency spectrum. The twenty-first century engineers know by now that space is the next frontier due to the omnipresence of spatial spectrum modes, which refocuses the programs towards a Stieltjes transform theory.

We sincerely hope that this book will inspire students, teachers, and engineers, and answer their present and future problems.

Romain Couillet and Mérouane Debbah
Acknowledgments

This book is the fruit of many years of the authors’ involvement in the field of random matrix theory for wireless communications. This topic, which has gained increasing interest in the last decade, was brought to light in the telecommunication community in particular through the work of Stephen Hanly, Ralf Muller, Shlomo Shamai, Emre Telatar, David Tse, Antonia Tulino, and Sergio Verdú, among others. It then rapidly grew into a joint research framework gathering both telecommunication engineers and mathematicians, among which Zhidong Bai, Vyacheslav L. Girko, Leonid Pastur, and Jack W. Silverstein.

The authors are especially indebted to Prof. Silverstein for the agreeable time spent discussing random matrix matters. Prof. Silverstein has a very insightful approach to random matrices, which it was a delight to share with him. The general point of view taken in this book is mostly influenced by Prof. Silverstein’s methodology. The authors are also grateful to the many colleagues working in this field whose knowledge and wisdom about applied random matrix theory contributed significantly to its current popularity and elegance. This book gathers many of their results and intends above all to deliver to the readers this simplified approach to applied random matrix theory. The colleagues involved in long and exciting discussions as well as collaborative works are Florent Benaych-Georges, Pascal Bianchi, Laura Cottatellucci, Maxime Guillaud, Walid Hachem, Philippe Loubaton, Mylene Ma da, Xavier Mestre, Aris Moustakas, Ralf Muller, Jamal Najim, and Øyvind Ryan.

Regarding the book manuscript itself, the authors would also like to sincerely thank the anonymous reviewers for their wise comments which contributed to improve substantially the overall quality of the final book and more importantly the few people who dedicated a long time to thoroughly review the successive drafts and who often came up with inspiring remarks. Among the latter are David Gregoratti, Jakob Hoydis, Xavier Mestre, and Sebastian Wagner.

The success of this book relies in a large part on these people.
<table>
<thead>
<tr>
<th>Acronym</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>AWGN</td>
<td>additive white Gaussian noise</td>
</tr>
<tr>
<td>BC</td>
<td>broadcast channel</td>
</tr>
<tr>
<td>BPSK</td>
<td>binary pulse shift keying</td>
</tr>
<tr>
<td>CDMA</td>
<td>code division multiple access</td>
</tr>
<tr>
<td>CI</td>
<td>channel inversion</td>
</tr>
<tr>
<td>CSI</td>
<td>channel state information</td>
</tr>
<tr>
<td>CSIR</td>
<td>channel state information at receiver</td>
</tr>
<tr>
<td>CSIT</td>
<td>channel state information at transmitter</td>
</tr>
<tr>
<td>d.f.</td>
<td>distribution function</td>
</tr>
<tr>
<td>DPC</td>
<td>dirty paper coding</td>
</tr>
<tr>
<td>e.s.d.</td>
<td>empirical spectral distribution</td>
</tr>
<tr>
<td>FAR</td>
<td>false alarm rate</td>
</tr>
<tr>
<td>GLRT</td>
<td>generalized likelihood ratio test</td>
</tr>
<tr>
<td>GOE</td>
<td>Gaussian orthogonal ensemble</td>
</tr>
<tr>
<td>GSE</td>
<td>Gaussian symplectic ensemble</td>
</tr>
<tr>
<td>GUE</td>
<td>Gaussian unitary ensemble</td>
</tr>
<tr>
<td>i.i.d.</td>
<td>independent and identically distributed</td>
</tr>
<tr>
<td>l.s.d.</td>
<td>limit spectral distribution</td>
</tr>
<tr>
<td>MAC</td>
<td>multiple access channel</td>
</tr>
<tr>
<td>MF</td>
<td>matched-filter</td>
</tr>
<tr>
<td>MIMO</td>
<td>multiple input multiple output</td>
</tr>
<tr>
<td>MISO</td>
<td>multiple input single output</td>
</tr>
<tr>
<td>ML</td>
<td>maximum likelihood</td>
</tr>
<tr>
<td>LMMSE</td>
<td>linear minimum mean square error</td>
</tr>
<tr>
<td>MMSE</td>
<td>minimum mean square error</td>
</tr>
<tr>
<td>MMSE-SIC</td>
<td>MMSE and successive interference cancellation</td>
</tr>
<tr>
<td>MSE</td>
<td>mean square error</td>
</tr>
<tr>
<td>MUSIC</td>
<td>multiple signal classification</td>
</tr>
<tr>
<td>NMSE</td>
<td>normalized mean square error</td>
</tr>
<tr>
<td>OFDM</td>
<td>orthogonal frequency division multiplexing</td>
</tr>
<tr>
<td>Acronyms</td>
<td>Description</td>
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<tr>
<td>-----------</td>
<td>--------------------------------------------------</td>
</tr>
<tr>
<td>OFDMA</td>
<td>orthogonal frequency division multiple access</td>
</tr>
<tr>
<td>p.d.f.</td>
<td>probability density function</td>
</tr>
<tr>
<td>QAM</td>
<td>quadrature amplitude modulation</td>
</tr>
<tr>
<td>QPSK</td>
<td>quadrature pulse shift keying</td>
</tr>
<tr>
<td>ROC</td>
<td>receiver operating characteristic</td>
</tr>
<tr>
<td>RZF</td>
<td>regularized zero-forcing</td>
</tr>
<tr>
<td>SINR</td>
<td>signal-to-interference plus noise ratio</td>
</tr>
<tr>
<td>SISO</td>
<td>single input single output</td>
</tr>
<tr>
<td>SNR</td>
<td>signal-to-noise ratio</td>
</tr>
<tr>
<td>TDMA</td>
<td>time division multiple access</td>
</tr>
<tr>
<td>ZF</td>
<td>zero-forcing</td>
</tr>
</tbody>
</table>
Notation

Linear algebra

\( \mathbf{X} \) \hspace{1cm} Matrix
\( I_N \) \hspace{1cm} Identity matrix of size \( N \times N \)
\( X_{ij} \) \hspace{1cm} Entry \((i, j)\) of matrix \( \mathbf{X} \) (unless otherwise stated)
\((X)_{ij}\) \hspace{1cm} Entry \((i, j)\) of matrix \( \mathbf{X} \)
\([X]_{ij}\) \hspace{1cm} Entry \((i, j)\) of matrix \( \mathbf{X} \)
\( f(i, j) \) \hspace{1cm} Matrix with \((i, j)\) entry \( f(i, j) \)
\((X_{ij})_{i,j} \) \hspace{1cm} Matrix with \((i, j)\) entry \( X_{ij} \)
\( \mathbf{x} \) \hspace{1cm} Vector (column by default)
\( x \) \hspace{1cm} Vector of the complex conjugates of the entries of \( \mathbf{x} \)
\( x_i \) \hspace{1cm} Entry \( i \) of vector \( \mathbf{x} \)
\( P^X \) \hspace{1cm} Empirical spectral distribution of the Hermitian \( \mathbf{X} \)
\( \mathbf{X}^\mathsf{T} \) \hspace{1cm} Transpose of \( \mathbf{X} \)
\( \mathbf{X}^\mathsf{H} \) \hspace{1cm} Hermitian transpose of \( \mathbf{X} \)
\( \text{tr} \mathbf{X} \) \hspace{1cm} Trace of \( \mathbf{X} \)
\( \det \mathbf{X} \) \hspace{1cm} Determinant of \( \mathbf{X} \)
\( \text{rank}(\mathbf{X}) \) \hspace{1cm} Rank of \( \mathbf{X} \)
\( (\mathbf{X}) \) \hspace{1cm} Vandermonde determinant of \( \mathbf{X} \)
\( \|\mathbf{X}\| \) \hspace{1cm} Spectral norm of the Hermitian matrix \( \mathbf{X} \)
\( \text{diag}(x_1, \ldots, x_n) \) \hspace{1cm} Diagonal matrix with \((i, i)\) entry \( x_i \)
\( \ker(\mathbf{A}) \) \hspace{1cm} Null space of the matrix \( \mathbf{A} \), \( \ker(\mathbf{A}) = \{ \mathbf{x} | \mathbf{A}\mathbf{x} = 0 \} \)
\( \text{span}(\mathbf{A}) \) \hspace{1cm} Subspace generated by the columns of the matrix \( \mathbf{A} \)

Real and complex analysis

\( \mathbb{N} \) \hspace{1cm} The space of natural numbers
\( \mathbb{R} \) \hspace{1cm} The space of real numbers
\( \mathbb{C} \) \hspace{1cm} The space of complex numbers
\( \mathbb{A} \) \hspace{1cm} The space \( \mathbb{A} = 0 \)
\( x^+ \) \hspace{1cm} Right-limit of the real \( x \)
\( x^- \) \hspace{1cm} Left-limit of the real \( x \)
Notation

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(x)^+$</td>
<td>For $x \in \mathbb{R}$, $\max(x, 0)$</td>
</tr>
<tr>
<td>$\text{sgn}(x)$</td>
<td>Sign of the real $x$</td>
</tr>
<tr>
<td>$[z]$</td>
<td>Real part of $z$</td>
</tr>
<tr>
<td>$[z]$</td>
<td>Imaginary part of $z$</td>
</tr>
<tr>
<td>$z^*$</td>
<td>Complex conjugate of $z$</td>
</tr>
<tr>
<td>$i$</td>
<td>Square root of $-1$ with positive imaginary part</td>
</tr>
<tr>
<td>$f'(x)$</td>
<td>First derivative of the function $f$</td>
</tr>
<tr>
<td>$f''(x)$</td>
<td>Second derivative of the function $f$</td>
</tr>
<tr>
<td>$f'''(x)$</td>
<td>Third derivative of the function $f$</td>
</tr>
<tr>
<td>$f^{(p)}(x)$</td>
<td>Derivative of order $p$ of the function $f$</td>
</tr>
<tr>
<td>$|f|$</td>
<td>Norm of a function $f = \sup_x f(x)$</td>
</tr>
<tr>
<td>$1_A(x)$</td>
<td>Indicator function of the set $A$</td>
</tr>
<tr>
<td>$1_A(x) = 1$ if $x \in A$, $1_A(x) = 0$ otherwise</td>
<td></td>
</tr>
<tr>
<td>$(x)$</td>
<td>Dirac delta function, $(x) = 1_0(x)$</td>
</tr>
<tr>
<td>$(x A)$</td>
<td>Convex indicator function $(x A) = 1$ if $x \in A$, $(x A) = 0$ otherwise</td>
</tr>
<tr>
<td>$\text{Supp}(F)$</td>
<td>Support of the distribution function $F$</td>
</tr>
<tr>
<td>$x_1, x_2$</td>
<td>Series of general term $x_n$</td>
</tr>
<tr>
<td>$x_n = o(y_n)$</td>
<td>Upon existence, $x_n y_n \to 0$ as $n$</td>
</tr>
<tr>
<td>$x_n = O(y_n)$</td>
<td>There exists $K$, such that $x_n K y_n$ for all $n$</td>
</tr>
<tr>
<td>$n \sim c$</td>
<td>As $n \to \infty$, $n \sim c$</td>
</tr>
<tr>
<td>$W(z)$</td>
<td>Lambert-W function satisfying $W(z) e^{W(z)} = z$</td>
</tr>
<tr>
<td>$\text{Ai}(x)$</td>
<td>Airy function</td>
</tr>
<tr>
<td>$(x)$</td>
<td>Gamma function, $(n) = (n - 1)!$ for $n$ integer</td>
</tr>
</tbody>
</table>

Probability theory

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\Omega, \mathcal{F}, P)$</td>
<td>Probability space with $\sigma$-field $\mathcal{F}$ and measure $P$</td>
</tr>
<tr>
<td>$P_X(x)$</td>
<td>Density of the random variable $X$</td>
</tr>
<tr>
<td>$p_X(x)$</td>
<td>Density of the scalar random variable $X$</td>
</tr>
<tr>
<td>$P_{X_1}(x)$</td>
<td>Unordered density of the random variable $X_1$</td>
</tr>
<tr>
<td>$P_{(X_1)}(x)$</td>
<td>Ordered density of the random variable $X_1$</td>
</tr>
<tr>
<td>$x$</td>
<td>Probability measure of $X$, $X(A) = P(X(A))$</td>
</tr>
<tr>
<td>$X$</td>
<td>Probability distribution of the eigenvalues of $X$</td>
</tr>
<tr>
<td>$P_X(x)$</td>
<td>Probability distribution associated with the l.s.d. of $X$</td>
</tr>
<tr>
<td>$F_X(x)$</td>
<td>Distribution function of $X$ (real), $F_X(x) = x({x})$</td>
</tr>
<tr>
<td>$E[X]$</td>
<td>Expectation of $X$, $E[X] = X(\cdot)dx$</td>
</tr>
<tr>
<td>Notation</td>
<td>Definition</td>
</tr>
<tr>
<td>----------</td>
<td>------------</td>
</tr>
<tr>
<td>$E[f(X)]$</td>
<td>Expectation of $f(X)$, $E[f(X)] = \int f(X) , d\omega$</td>
</tr>
<tr>
<td>$\text{var}(X)$</td>
<td>Variance of $X$, $\text{var}(X) = E[X^2] - E[X]^2$</td>
</tr>
<tr>
<td>$X \sim \mathcal{L}$</td>
<td>$X$ is a random variable with density $\mathcal{L}$</td>
</tr>
<tr>
<td>$\mathcal{N}(\mu, \Sigma)$</td>
<td>Complex Gaussian distribution of mean $\mu$ and covariance $\Sigma$</td>
</tr>
<tr>
<td>$\mathcal{W}_n(n, R)$</td>
<td>Real zero mean Wishart distribution with $n$ degrees of freedom and covariance $R$</td>
</tr>
<tr>
<td>$e\mathcal{W}_n(n, R)$</td>
<td>Complex zero mean Wishart distribution with $n$ degrees of freedom and covariance $R$</td>
</tr>
<tr>
<td>$Q(x)$</td>
<td>Gaussian $Q$-function, $Q(x) = P(X &gt; x)$, $X \sim \mathcal{N}(0, 1)$</td>
</tr>
<tr>
<td>$F^+$</td>
<td>Conjugate Tracy–Widom distribution function</td>
</tr>
<tr>
<td>$F_n$</td>
<td>Weak convergence of the d.f. series $F_1, F_2, \ldots$ to $F$</td>
</tr>
<tr>
<td>$X_n$</td>
<td>Weak convergence of the series $X_1, X_2, \ldots$ to the random $X$</td>
</tr>
</tbody>
</table>

**Random Matrix Theory**

$m_F(z)$ | Stieltjes transform of the function $F$
---|---
$m_{X}(z)$ | Stieltjes transform of the eigenvalue distribution of $X$
$v_F(z)$ | Shannon transform of the function $F$
$v_X(z)$ | Shannon transform of the eigenvalue distribution of $X$
$R_F(z)$ | R transform of the function $F$
$R_X(z)$ | R transform of the eigenvalue distribution of $X$
$S_F(z)$ | S transform of the function $F$
$S_X(z)$ | S transform of the eigenvalue distribution of $X$
$f(z)$ | -transform of the function $F$
$x(z)$ | -transform of the eigenvalue distribution of $X$
$ar{f}(z)$ | -transform of the function $F$
$ar{x}(z)$ | -transform of the eigenvalue distribution of $X$

**Topology**

$A^c$ | Complement of the set $A$
---|---
$\#A$ | Cardinality of the discrete set $A$
$A \oplus B$ | Direct sum of the spaces $A$ and $B$
$1_{n} \oplus A_i$ | Direct sum of the spaces $A_i$, $1 \leq i \leq n$
$x \ A$ Norm of the orthogonal projection of $x$ on the space $A$

Miscellaneous

$x \ y \ x$ is defined as $y$

$\text{sgn}(\ )$ Signature (or parity) of the permutation $\ , \text{sgn}(\ )$ 1 1