Introduction

Quantum fields and *quantization* are concepts that come from quantum physics, the most intriguing physical theory developed in the twentieth century. In our work we would like to describe in a coherent and comprehensive way basic aspects of their mathematical structure.

Most of our work is devoted to the simplest kinds of quantum fields and of quantization. We will mostly discuss mathematical aspects of *free* quantum fields. We will consider the quantization only on *linear* phase spaces. The reader will see that even within such a restricted scope the subject is rich, involves many concepts and has important applications, both to quantum theory and to pure mathematics.

A distinguished role in our work will be played by representations of the *canonical commutation* and *anti-commutation relations*. Let us briefly discuss the origin and the meaning of these concepts.

Let us start with *canonical commutation relations*, abbreviated commonly as the *CCR*. Since the early days of quantum mechanics it has been noted that the *position operator* x and the *momentum operator* $D = -\mathrm{i}\nabla$ satisfy the following commutation relation:

$$[x, D] = \mathrm{i}1\!\!1. \tag{1}$$

If we set $a^* = \frac{1}{\sqrt{2}}(x - \mathrm{i}D)$, $a = \frac{1}{\sqrt{2}}(x + \mathrm{i}D)$, called the *bosonic creation* and *annihilation operators*, we obtain

$$[a, a^*] = 1\!\!1. \tag{2}$$

We easily see that (1) is equivalent to (2).

Strictly speaking, the identities (1) and (2) are ill defined because it is not clear how to interpret the commutator of unbounded operators. Weyl proposed replacing (1) by

$$\mathrm{e}^{\mathrm{i}\eta x}\mathrm{e}^{\mathrm{i}qD} = \mathrm{e}^{-\mathrm{i}q\eta}\mathrm{e}^{\mathrm{i}qD}\mathrm{e}^{\mathrm{i}\eta x}, \ \ \eta, q \in \mathbb{R}, \tag{3}$$

which has a clear mathematical meaning. (1) is often called the *CCR in the Heisenberg form* and (3) *in the Weyl form*.

It is natural to ask whether the commutation relations (1) determine the operators x and D uniquely up to unitary equivalence. If we assume that we are given two self-adjoint operators x and D acting irreducibly on a Hilbert space and satisfying (3), then the answer is positive, as proven by Stone and von Neumann.

Relations (1) and (2) involve a classical system with one degree of freedom. One can also generalize the CCR to systems with many degrees of freedom. Systems with a finite number of degrees of freedom appear e.g. in the quantum mechanical description of atoms or molecules, while systems with an infinite number of degrees of freedom are typical for quantum many-body physics and quantum field theory.

In the case of many degrees of freedom it is often useful to use a more abstract setting for the CCR. One can consider a family of self-adjoint operators $\phi_1, \phi_2, \dots$ satisfying the relations

$$[\phi_j, \phi_k] = \mathrm{i}\omega_{jk}\mathbb{1}, \tag{4}$$

where ω_{jk} is an anti-symmetric matrix. Alternatively, one can consider the Weyl (exponentiated) form of (4) satisfied by the so-called *Weyl operators* $\exp\big(\mathrm{i}\sum_i y_i\phi_i\big)$, where y_i are real coefficients.

A typical example of CCR with many, possibly an infinite number of, degrees of freedom appears in the context of second quantization, where one introduces *bosonic creation* and *annihilation operators* a_i^*, a_j satisfying an extension of (2):

$$\begin{aligned}&[a_i, a_j] = [a_i^*, a_j^*] = 0,\\&[a_i, a_j^*] = \delta_{ij}\mathbb{1}.\end{aligned} \tag{5}$$

The Stone–von Neumann theorem can be extended to the case of regular CCR representations for a finite-dimensional symplectic matrix ω_{jk}. Note that in this case the relations (4) are invariant with respect to the symplectic group. This invariance is implemented by a projective unitary representation of the symplectic group. It can be expressed in terms of a representation of the two-fold covering of the symplectic group – the so-called *metaplectic representation*.

Symplectic invariance is also a characteristic feature of classical mechanics. In fact, one usually assumes that the phase space of a classical system is a symplectic manifold and its symmetries, including the time evolution, are described by symplectic transformations. One of the main aspects of the correspondence principle is the fact that the symplectic invariance plays an important role both in classical mechanics and in the context of canonical commutation relations.

The symplectic invariance of the CCR plays an important role in many problems of quantum theory and of partial differential equations. An interesting – and historically perhaps the first – non-trivial application of this invariance is due to Bogoliubov, who used it in the theory of superfluidity of the Bose gas; see Bogoliubov (1947b). Since then, applications of symplectic transformations to the study of bosonic systems often go in the physics literature under the name *Bogoliubov method*.

Let us now discuss the *canonical anti-commutation relations*, abbreviated commonly as the *CAR*. They are closely related to the so-called *Clifford relations*, which appeared in mathematics before quantum theory, in Clifford (1878). We say that operators $\phi_1, \dots, \phi_n$ satisfy Clifford relations if

$$[\phi_i, \phi_j]_+ = 2g_{ij}\mathbb{1}, \tag{6}$$

where g_{ij} is a symmetric non-degenerate matrix and $[A, B]_+ := AB + BA$ denotes the anti-commutator of A and B. It is not difficult to show that if the representation (6) is irreducible, then it is unique up to a unitary equivalence for n even, and there are two inequivalent representations for n odd.

In quantum physics, CAR appeared in the description of fermions. If $a_1^*, \dots, a_m^*$ are *fermionic creation* and $a_1, \dots, a_m$ *fermionic annihilation operators*, then they satisfy

$$[a_i^*, a_j^*]_+ = 0, \quad [a_i, a_j]_+ = 0, \quad [a_i^*, a_j]_+ = \delta_{ij}\mathbb{1}.$$

If we set $\phi_{2j-1} := a_j^* + a_j$, $\phi_{2j} := \frac{1}{\mathrm{i}}(a_j^* - a_j)$, then they satisfy the relations (6) with $n = 2m$ and $g_{ij} = \delta_{ij}$. Besides, the operators ϕ_i are then self-adjoint.

Another family of operators satisfying the CAR in quantum physics are the *Pauli matrices* used in the description of spin $\frac{1}{2}$ particles. The *Dirac matrices* also satisfy Clifford relations, with g_{ij} equal to the Minkowski metric tensor.

Clearly, the relations (6) with $g_{ij} = \delta_{ij}$ are preserved by orthogonal transformations applied to $(\phi_1, \dots, \phi_n)$. The orthogonal invariance of CAR is implemented by a projective unitary representation. It can be also expressed in terms of a representation of the double covering of the orthogonal group, called the *Pin group*.

The orthogonal invariance of CAR relations appears in many disguises in algebra, differential geometry and quantum physics. In quantum physics its applications are again often called the *Bogoliubov method*. A particularly interesting application of this method can be found in the theory of superconductivity and goes back to Bogoliubov (1958).

The notion of CCR and CAR representations is quite elementary in the case of a finite number of degrees of freedom. It becomes much deeper for an infinite number of degrees of freedom. In this case there exist many inequivalent CCR and CAR representations, a fact that was not recognized before the 1950s.

The most commonly used CCR and CAR representations are the so-called *Fock representations*, acting on *bosonic,* resp. *fermionic Fock spaces*. These spaces have a distinguished vector Ω called the *vacuum*, killed by annihilation operators and cyclic with respect to creation operators.

In the case of an infinite number of degrees of freedom, the symplectic or orthogonal invariance of representations of CCR, resp. CAR becomes much more subtle. In particular, not every symplectic, resp. orthogonal transformation is unitarily implementable on the Fock space. The *Shale*, resp. *Shale–Stinespring theorem* say that implementable symplectic, resp. orthogonal transformations belong to a relatively small group $Sp_{\mathrm{j}}(\mathcal{Y})$, resp. $O_{\mathrm{j}}(\mathcal{Y})$. Other interesting objects in the case of an infinite number of degrees of freedom are the analogs of the metaplectic and Pin representation.

CCR and CAR representations provide a convenient setting to describe various forms of *quantization*. By a quantization we usually mean a map that transforms a function on a classical phase space into an operator and has some good properties. Of course, this is not a precise definition – actually, there seems to be no generally accepted definition of the term "quantization". Clearly, some quantizations are better and more useful than others.

We describe a number of the most important and useful quantizations. In the case of CCR, they include the *Weyl, Wick, anti-Wick,* x, D*- and* D, x*-quantizations*. In the case of CAR, we discuss the *anti-symmetric, Wick and anti-Wick quantizations*. Among these quantizations, the Weyl, resp. the anti-symmetric quantization play a distinguished role, since they preserve the underlying symmetry of the CCR, resp. CAR – the symplectic, resp. orthogonal group. However, they are not very useful for an infinite number of degrees of freedom, in which case the Wick quantization is much better behaved. The x, D-quantization is a favorite tool in the *microlocal analysis* of partial differential equations.

The non-uniqueness of CCR or CAR representations for an infinite number of degrees of freedom is a motivation for adopting a purely algebraic point of view, without considering a particular representation. This leads to the use of operator algebras in the description of the CCR and CAR. This is easily done in the case of the CAR, where there exists an obvious candidate for the *CAR* C^**-algebra* corresponding to a given Euclidean space. This algebra belongs to the well-known class of *uniformly hyper-finite algebras*, the so-called UHF(2^∞) *algebra*. We also have a natural *CAR* W^**-algebra*. It has the structure of the well-known *injective type* II_1 *factor*.

In the case of the CCR, the choice of the corresponding C^*-algebra is less obvious. The most popular choice seems to be the C^*-algebra generated by the Weyl operators, called sometimes the *Weyl CCR algebra*. One can, however, argue that the Weyl CCR algebra is not very physical and that there are other more natural choices of the C^*-algebra of CCR.

Essentially all CCR and CAR representations used in practical computations belong to the so-called *quasi-free representations*. They appear naturally, e.g. in the description of thermal states of the Bose and Fermi gas. They have interesting mathematical properties from the point of view of operator algebras. In particular, they provide interesting and physically well motivated examples of *factors of type II and III*. They also give good illustrations for the *Tomita–Takesaki modular theory* and for the so-called *standard form of a* W^**-algebra*.

The formalism of CCR and CAR representations gives a convenient language for many useful aspects of quantum field theory. This is especially true in the case of free quantum fields, where representations of the CCR and CAR constitute, in one form or another, a part of the standard language. More or less explicitly they are used in all textbooks on quantum field theory. Usually the authors first discuss quantum fields *classically*. In other words, they just describe algebraic relations satisfied by the fields without specifying their representation. In relativistic quantum field theory these relations are usually derived from some form of classical field equations, like the *Klein–Gordon equation* for bosonic fields and the *Dirac equation* for fermionic fields.

In the next step a representation of CCR or CAR relations on a Hilbert space is introduced. The choice of this representation usually depends on the dynamics and the temperature. At the *zero temperature*, it is usually the Fock representation determined by the requirement that the dynamics should be implemented by a self-adjoint, bounded from below *Hamiltonian*. At *positive temperatures* one usually chooses the *GNS representation* given by an appropriate *KMS state*.

Another related topic is the problem of the unitary implementability of various symmetries of a given theory, such as for example Lorentz transformations in relativistic models. If the generator of the dynamics depends on time, one can also ask whether there exists a time-dependent Hamiltonian that implements the dynamics.

Models of quantum field theory that appear in realistic applications are usually *interacting*, meaning that they cannot be derived from a linear transformation of the underlying phase space. Interacting models are usually described as formal perturbations of free ones. Various terms in perturbation expansions are graphically depicted with *diagrams*. The diagrammatic method is a standard tool for the perturbative computation of various physical quantities.

In the 1950s, mathematical physicists started to apply methods from spectral theory to construct rigorously interacting quantum field theory models. After a while, this subject became dominated by the so-called *Euclidean methods*. The main idea of these methods is to make the real time variable purely imaginary. The Euclidean point of view is nowadays often used as the basic one, at both zero and positive temperature.

Many concepts that we discuss in our work originated in quantum physics and have a strong physical motivation. We believe that our work (or at least some of its parts) can be useful in teaching some chapters of quantum physics. In fact, we believe that the mathematical style is often better suited to explaining some concepts of quantum theory than the style found in many physics textbooks.

Note, however, that the reader does not have to know physics at all in order to follow and, it is hoped, to appreciate our work. In our opinion, essentially all the concepts and results that we discuss are natural and appealing from the point of view of pure mathematics.

We expect that the reader is familiar and comfortable with a relatively broad spectrum of mathematics. We freely use various basic facts and concepts from linear algebra, real analysis, the theory of operators on Hilbert spaces, operator algebras and measure theory.

The theory of the CCR and CAR involves a large number of concepts coming from algebra, analysis and physics. Therefore, it is not surprising that the literature about this subject is very scattered, and uses various conventions, notations and terminology.

We have made an effort to introduce terminology and notation that is as consistent and transparent as possible. In particular, we tried to stress close analogies between the CCR and CAR. Therefore, we have tried to present both formalisms in a possibly parallel way. We make an effort to present many topics in their greatest mathematical generality. We believe that this way of presentation is efficient, especially for mathematically mature readers.

The literature devoted to topics contained in our book is quite large. Let us mention some of the monographs. The exposition of the C^*-algebraic approach to the CCR and CAR can be found in Bratteli–Robinson (1996). This monograph also provides extensive historical remarks. One could also consult an older monograph, Emch (1972). Modern exposition of the mathematical formalism of second quantization can be also found e.g. in Glimm–Jaffe (1987) and Baez–Segal–Zhou (1991). We would also like to mention the book by Neretin (1996), which describes infinite-dimensional metaplectic and Pin groups, and review articles by Varilly–Gracia-Bondia (1992, 1994). A very comprehensive article devoted to CAR C^*-algebras was written by Araki (1987). Introductions to Clifford algebras can be found in Lawson–Michelson (1989) and Trautman (2006).

The book can be naturally divided into four parts.

(1) Chapters 1, 2, 3, 4, 5 6 and 7 are mostly collections of basic mathematical facts and definitions, which we use in the remaining part of our work. Not all the mathematical formalism presented in these chapters is of equal importance for the main topic of work. Perhaps, most readers are advised to skip these chapters on the first reading, consulting them when needed.
(2) Chapters 8, 9, 10 and 11 are devoted to the canonical commutation relations. We discuss in particular various kinds of quantization of bosonic systems and the bosonic Fock representation. We describe the *metaplectic group* and its various infinite-dimensional generalizations.
(3) In Chaps. 12, 13, 14, 15 and 16 we develop the theory of canonical anticommutation relations. It is to a large extent parallel to the previous chapters devoted to the CCR. We discuss, in particular, the fermionic Fock representation. As compared with the bosonic case, a bigger role is played by operator algebras. We give also a brief introduction to Clifford relations for an arbitrary signature. We discuss the *Pin* and *Spin groups* and their various infinite-dimensional generalizations.
(4) The common theme of the remaining part of the book, that is, Chaps. 17, 18, 19, 20, 21 and 22, is the concept of quantum dynamics – one-parameter unitary groups that describe the evolution of quantum systems. In all these chapters we treat the bosonic and fermionic cases in a parallel way, except for Chaps. 21 and 22, where we restrict ourselves to bosons.

In Chap. 17 we discuss *quasi-free states*. These usually arise as KMS states for a physical system equipped with a free dynamics. In Chaps. 18 and 19 we study quantization of free fields, first in the abstract context, then on a (possibly, curved) space-time. Chapters 20, 21 and 22 are devoted to interacting quantum field theory. In Chap. 20 we discuss in an abstract setting the method of *Feynman diagrams*. In Chap. 21 we describe the *Euclidean method*, used to construct interacting bosonic theories. In Chap. 22 we apply Euclidean methods to construct the so-called *space-cutoff* $P(\varphi)_2$ *model*.

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1
Vector spaces

In this chapter we fix our terminology and notation, mostly related to (real and complex) linear algebra. We will consider only algebraic properties. Infinite-dimensional vector spaces will not be equipped with any topology.

Let us stress that using precise terminology and notation concerning linear algebra is very useful in describing various aspects of quantization and quantum fields. Even though the material of this chapter is elementary, the terminology and notation introduced in this chapter will play an important role throughout our work. In particular we should draw the reader's attention to the notion of the complex conjugate space (Subsect. 1.2.3), and of the holomorphic and anti-holomorphic subspaces (Subsect. 1.3.6).

Throughout the book $\mathbb{K}$ will denote either the field $\mathbb{R}$ or $\mathbb{C}$, all vector spaces being either real or complex, unless specified otherwise.

1.1 Elementary linear algebra

The material of this section is well known and elementary. Among other things, we discuss four basic kinds of structures, which will serve as the starting point for quantization:

(1) Symplectic spaces – classical phase spaces of neutral bosons,
(2) Euclidean spaces – classical phase spaces of neutral fermions,
(3) Charged symplectic spaces – classical phase spaces of charged bosons,
(4) Unitary spaces – classical phase spaces of charged fermions.

Throughout the section, $\mathcal{Y}, \mathcal{Y}_1, \mathcal{Y}_2, \mathcal{W}$ are vector spaces over $\mathbb{K}$.

1.1.1 Vector spaces and linear operators

Definition 1.1 *If* $\mathcal{U} \subset \mathcal{Y}$*, then* Span $\mathcal{U}$ *denotes the space of finite linear combinations of elements of* $\mathcal{U}$*.*

Definition 1.2 $\mathcal{Y}_1 \oplus \mathcal{Y}_2$ *denotes the* external direct sum of $\mathcal{Y}_1$ and $\mathcal{Y}_2$*, that is, the Cartesian product* $\mathcal{Y}_1 \times \mathcal{Y}_2$ *equipped with its vector space structure. If* $\mathcal{Y}_1$*,* $\mathcal{Y}_2$ *are subspaces of a vector space* $\mathcal{Y}$ *and* $\mathcal{Y}_1 \cap \mathcal{Y}_2 = \{0\}$*, then the same notation* $\mathcal{Y}_1 \oplus \mathcal{Y}_2$ *stands for the* internal direct sum of $\mathcal{Y}_1$ and $\mathcal{Y}_2$*, that is,* $\mathcal{Y}_1 + \mathcal{Y}_2$ *(which is a subspace of* $\mathcal{Y}$*).*

Definition 1.3 *$L(\mathcal{Y},\mathcal{W})$ denotes the space of linear maps from $\mathcal{Y}$ to $\mathcal{W}$. We set $L(\mathcal{Y}) := L(\mathcal{Y},\mathcal{Y})$.*

Definition 1.4 *$L^{\rm fd}(\mathcal{Y},\mathcal{W})$, resp. $L^{\rm fd}(\mathcal{Y})$ denote the space of* finite-dimensional *(or* finite rank) linear operators *in $L(\mathcal{Y},\mathcal{W})$, resp. $L(\mathcal{Y})$.*

Definition 1.5 *Let $a_i \in L(\mathcal{Y}_i,\mathcal{W})$, $i = 1,2$. We say that $a_1 \subset a_2$ if $\mathcal{Y}_1 \subset \mathcal{Y}_2$ and a_1 is the restriction of a_2 to $\mathcal{Y}_1$, that is, $a_2\big|_{\mathcal{Y}_1} = a_1$.*

Definition 1.6 *If $a \in L(\mathcal{Y},\mathcal{W})$, then* Ker a *denotes the* kernel *(or* null space) of a *and* Ran a *denotes its* range.

Definition 1.7 $1\!\!1_{\mathcal{Y}}$ *stands for the* identity on $\mathcal{Y}$.

1.1.2 2×2 *block matrices*

If $\mathcal{Y} = \mathcal{Y}_+ \oplus \mathcal{Y}_-$, every $r \in L(\mathcal{Y})$ can be written as a 2×2 block matrix. The following decomposition, possible if a is invertible, is often useful:

$$r = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1\!\!1 & 0 \\ ca^{-1} & 1\!\!1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & d - ca^{-1}b \end{bmatrix} \begin{bmatrix} 1\!\!1 & a^{-1}b \\ 0 & 1\!\!1 \end{bmatrix}. \tag{1.1}$$

Here are some expressions for the inverse of r:

$$r^{-1} = \begin{bmatrix} 1\!\!1 & -a^{-1}b \\ 0 & 1\!\!1 \end{bmatrix} \begin{bmatrix} a^{-1} & 0 \\ 0 & (d - ca^{-1}b)^{-1} \end{bmatrix} \begin{bmatrix} 1\!\!1 & 0 \\ -ca^{-1} & 1\!\!1 \end{bmatrix} \tag{1.2}$$

$$= \begin{bmatrix} (a - bd^{-1}c)^{-1} & (c - db^{-1}a)^{-1} \\ (b - ac^{-1}d)^{-1} & (d - ca^{-1}b)^{-1} \end{bmatrix}. \tag{1.3}$$

If $\mathcal{Y}$ is finite-dimensional, then, using the decomposition (1.1), we obtain the following formulas for the determinant:

$$\begin{aligned} \det r &= \det a \det(d - ca^{-1}b) \\ &= \det c \det b \det(ac^{-1}db^{-1} - 1\!\!1). \end{aligned} \tag{1.4}$$

1.1.3 Duality

Definition 1.8 *The* dual *of $\mathcal{Y}$, denoted by $\mathcal{Y}^{\#}$, is the space of* linear functionals on $\mathcal{Y}$. *Three kinds of notation for the action of $v \in \mathcal{Y}^{\#}$ on $y \in \mathcal{Y}$ will be used:*

(1) *the* bra–ket notation $\langle v|y\rangle = \langle y|v\rangle$,
(2) *the* simplified notation $v \cdot y = y \cdot v$,
(3) *the* functional notation $v(y)$.

There is a canonical injection $\mathcal{Y} \to \mathcal{Y}^{\#\#}$. We have $\mathcal{Y} = \mathcal{Y}^{\#\#}$ iff $\dim\mathcal{Y} < \infty$.

Definition 1.9 *If $y \in \mathcal{Y}$, we will sometimes write $|y\rangle$ for the operator*

$$\mathbb{K} \ni \lambda \mapsto |y\rangle\lambda := \lambda y \in \mathcal{Y}.$$

If $v \in \mathcal{Y}^{\#}$, we will sometimes write $\langle v|$ instead of v.

As an example of this notation, suppose that $y \in \mathcal{Y}$ and $v \in \mathcal{Y}^{\#}$ satisfy $\langle v|y\rangle = 1$. Then $|y\rangle\langle v|$ is the projection onto the space spanned by y along the kernel of v.

Definition 1.10 *Let $(e_1, \ldots, e_n)$ be a basis of a finite-dimensional space $\mathcal{Y}$. Then there exists a unique basis of $\mathcal{Y}^{\#}$, $(e^1, \ldots, e^n)$, called the* dual basis, *such that $\langle e^i|e_j\rangle = \delta^i_j$.*

1.1.4 Annihilator

Definition 1.11 *The* annihilator of $\mathcal{X} \subset \mathcal{Y}$ *is defined as*

$$\mathcal{X}^{\mathrm{an}} := \big\{v \in \mathcal{Y}^{\#} \ : \ \langle v|y\rangle = 0,\ y \in \mathcal{X}\big\}.$$

The pre-annihilator of $\mathcal{V} \subset \mathcal{Y}^{\#}$ *is defined as*

$$\mathcal{V}_{\mathrm{an}} := \big\{y \in \mathcal{Y} \ : \ \langle v|y\rangle = 0,\ v \in \mathcal{V}\big\}.$$

Note that

$$(\mathcal{X}^{\mathrm{an}})_{\mathrm{an}} = \mathrm{Span}\mathcal{X}, \quad (\mathcal{V}_{\mathrm{an}})^{\mathrm{an}} = \mathrm{Span}\mathcal{V}.$$

1.1.5 Transpose of an operator

Definition 1.12 *If $a \in L(\mathcal{Y}_1, \mathcal{Y}_2)$, then $a^{\#}$ will denote the* transpose of a, *that is, the operator in $L(\mathcal{Y}_2^{\#}, \mathcal{Y}_1^{\#})$ defined by*

$$\langle a^{\#} v|y\rangle := \langle v|ay\rangle, \quad v \in \mathcal{Y}_2^{\#}, \quad y \in \mathcal{Y}_1. \tag{1.5}$$

Note that a is bijective iff $a^{\#}$ is. We have $a^{\#\#} \in L(\mathcal{Y}_1^{\#\#}, \mathcal{Y}_2^{\#\#})$ and $a \subset a^{\#\#}$.

1.1.6 Dual pairs

Definition 1.13 *A* dual pair *is a pair $(\mathcal{V}, \mathcal{Y})$ of vector spaces equipped with a bilinear form*

$$(\mathcal{V}, \mathcal{Y}) \ni (v, y) \mapsto \langle v|y\rangle \in \mathbb{K}$$

such that

$$\langle v|y\rangle = 0,\ v \in \mathcal{V} \ \Rightarrow \ y = 0, \tag{1.6}$$

$$\langle v|y\rangle = 0,\ y \in \mathcal{Y} \ \Rightarrow \ v = 0. \tag{1.7}$$

Clearly, if $(\mathcal{V}, \mathcal{Y})$ is a dual pair, then so is $(\mathcal{Y}, \mathcal{V})$. If $\mathcal{Y}$ is finite-dimensional and $(\mathcal{V}, \mathcal{Y})$ is a dual pair, then $\mathcal{V}$ is naturally isomorphic to $\mathcal{Y}^{\#}$.

In general, $(\mathcal{V}, \mathcal{Y})$ is a dual pair iff $\mathcal{V}$ can be identified with a subspace of $\mathcal{Y}^{\#}$ (this automatically guarantees (1.7)) satisfying $\mathcal{V}_{\mathrm{an}} = \{0\}$ (this implies (1.6)).