1 Static Plastic Behaviour of Beams

1.1 Introduction

Many ductile materials which are used in engineering practice have a considerable reserve capacity beyond the initial yield condition. The uniaxial yield strain of mild steel, for example, is 0.001 approximately, whereas this material ruptures, in a standard static uniaxial tensile test, at an engineering strain of 0.3, approximately. This reserve strength may be utilised in a structural design to provide a more realistic estimate of the safety factor against failure for various extreme loads. Thus, the static plastic behaviour of structures has been studied extensively and is introduced in many textbooks.[1.1–1.11] An interested reader is referred to these textbooks for a deeper presentation of the subject than is possible in this book, which is concerned primarily with the influence of dynamic loadings. However, the methods of dynamic structural plasticity presented in this book owe a substantial debt to the theoretical foundation of static structural plasticity, which is, therefore, reviewed briefly in this chapter and the following one.

A considerable body of literature is available on the static behaviour of structures made from ductile materials which may be idealised as perfectly plastic. This simplification allows the principal characteristics and overall features of the structural response to be obtained fairly simply for many important practical cases. Moreover, the static collapse loads predicted by these simplified methods often provide good estimates of the corresponding experimental values. Indeed, the design codes in several industries now permit the use of plasticity theory for the design of various structures and components. The theoretical background of these methods, which were developed primarily to examine the static loading of structures made from perfectly plastic materials, are valuable for studies into the response of structures subjected to dynamic loads. Thus, this chapter and the next focus on the static behaviour of structures which are made from perfectly plastic materials.

The basic equations which govern the static behaviour of beams are introduced in the next section. The plastic collapse, or limit moment, is also derived in § 1.2 for a beam with a solid rectangular cross-section which is subjected to a pure bending moment. However, considerable effort is required sometimes to obtain the exact collapse load of a beam which is subjected to a more general form of loading. Thus, the lower and upper bound theorems of plastic collapse are proved in § 1.3. These theorems provide a simple yet rigorous procedure for bounding the exact plastic collapse load of a beam which is subjected to any form of external loading, as illustrated in § 1.4 to § 1.7 for several cases.

A heuristic approach is introduced in § 1.8 and used to obtain the exact static plastic collapse load of a partially loaded beam. Some experimental results are reported in § 1.9 and a few final remarks are given in § 1.10.

1.2 Basic Equations for Beams

Beams are defined as structural members having a length which is large compared with the corresponding width and depth. It is observed in this circumstance that the lateral, or transverse, shear stresses are small compared with the axial, or longitudinal, stresses. Moreover, it is reasonable to replace the actual force distribution across the depth of a beam by a lateral, or transverse, shear force $Q\,(Q = \int_A \sigma_{xz} \mathrm{d}A)$ and a bending moment $M\,(M = \int_A \sigma_x z\, \mathrm{d}A)$, as shown in Figure 1.1. The actual strain field is then described in terms of the curvature change of the longitudinal axis.† These assumptions lead to considerable simplifications in analyses, and are the usual ones which are incorporated in the engineering theory of elastic beams. It has been shown by Hodge(1.2) that these approximations are also acceptable for the behaviour of perfectly plastic beams.

The moment and lateral force equilibrium equations for the beam in Figure 1.1 are

$$\mathrm{d}M/\mathrm{d}x = Q \tag{1.1}$$

and

$$\mathrm{d}Q/\mathrm{d}x = -p, \tag{1.2}$$

respectively, when the response is time-independent and where p is the external load per unit length. The corresponding change in curvature of the longitudinal axis is

$$\kappa = -\mathrm{d}^2 w/\mathrm{d}x^2, \qquad (1.3)^\ddagger$$

provided $\mathrm{d}w/\mathrm{d}x \ll 1$.

Now consider a beam with a solid rectangular cross-section of breadth B and depth H, as shown in Figure 1.2(*a*). This beam is made from the elastic, perfectly plastic material in Figure 1.3 and is subjected to a pure bending moment M. Initially, the stress distribution across the depth of this beam is linear (see Figure 1.2(*b*)), so that the corresponding M–κ relation is also linear with a slope EI, as shown in

† The bending moment M produces a curvature change and is known as a generalised stress,(1.3) whereas the transverse shear force Q is a reaction since it does not produce any deformation of a beam.

‡ The curvature change is known as a generalised strain.(1.3) The product of generalised stress and the corresponding generalised strain rate gives a positive (or zero) energy dissipation rate (i.e., $M\dot{\kappa} \geq 0$, where $(\dot{\ })$ is a time derivative).

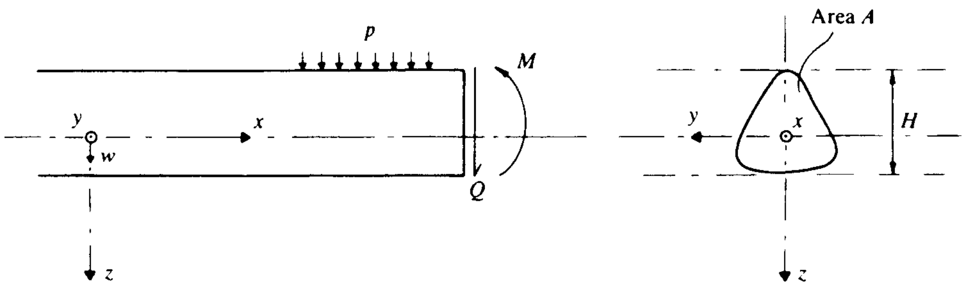


Figure 1.1. Notation for a beam.

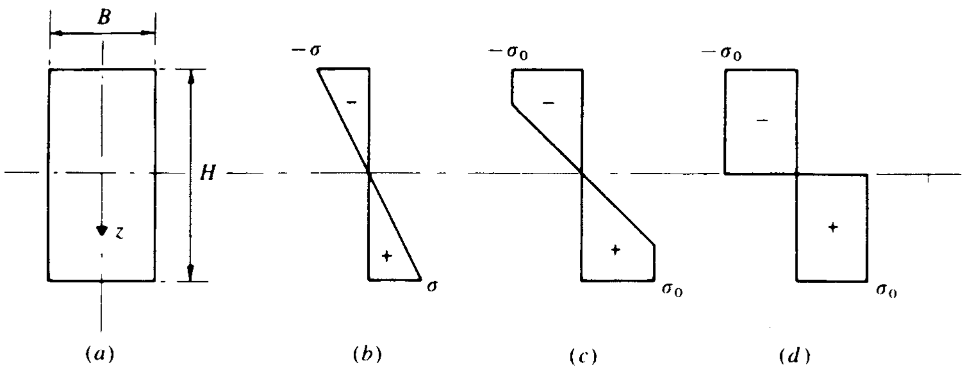


Figure 1.2. Development of the plastic zones in an elastic, perfectly plastic beam with a rectangular shaped cross-section and subjected to a pure bending moment: (*a*) rectangular cross-section; (*b*) elastic stress distribution; (*c*) elastic, perfectly plastic stress distribution; (*d*) fully plastic stress distribution. + and − denote tensile and compressive stresses, respectively, for a pure bending moment acting, as shown in Figure 1.1.

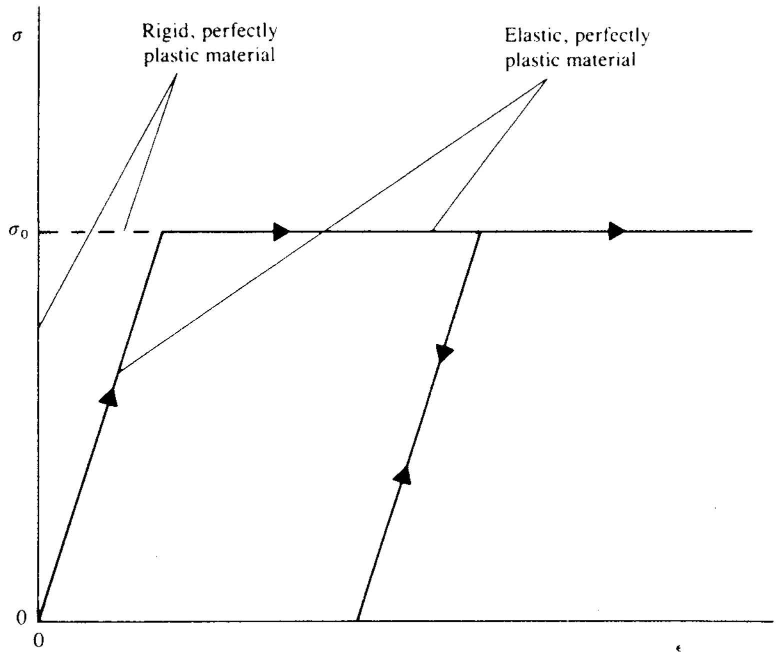


Figure 1.3. Elastic, perfectly plastic and rigid, perfectly plastic uniaxial stress–strain idealisations.

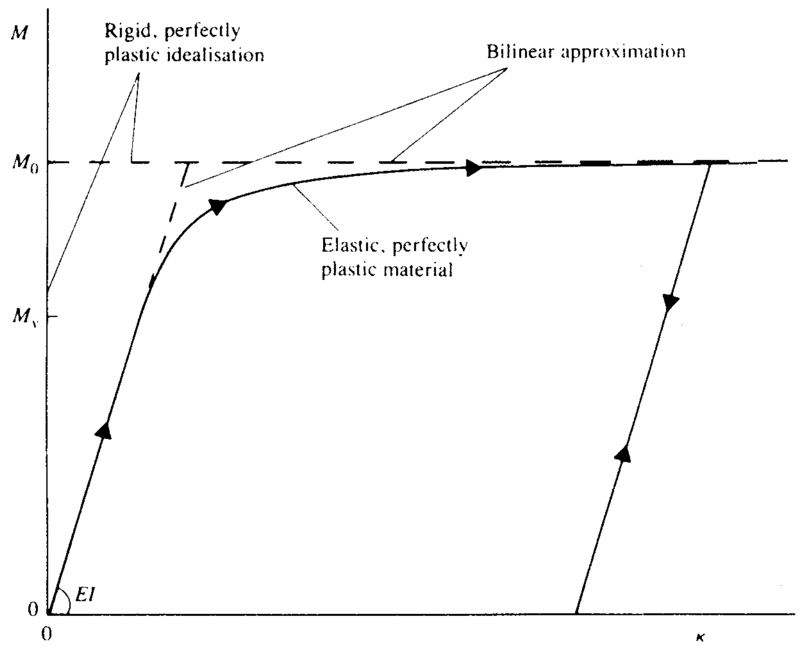


Figure 1.4. Moment–curvature characteristics for a beam with a rectangular cross-section.

Figure 1.4. (E is Young's modulus and I is the second moment of area for the cross-section.) If the applied bending moment is increased beyond the magnitude of the yield moment

$$M_y = 2I\sigma_0/H = \sigma_0 BH^2/6, \tag{1.4}$$ †

then yielding occurs in outer zones, as indicated in Figure 1.2(c), while the associated M–κ relation becomes non-linear. The applied bending moment can be increased further until the entire cross-section has yielded plastically and the strength of the beam has been exhausted, as shown in Figure 1.2(d). This maximum bending moment is known as the limit or collapse moment for the cross-section and can be written as

$$M_0 = (\sigma_0 BH/2)\,H/2 = \sigma_0 BH^2/4, \tag{1.5}$$ ‡

which follows from Figure 1.2(d). In order to simplify theoretical calculations on the plastic behaviour of beams with solid rectangular cross-sections, the moment–curvature relation is often replaced by the rigid, perfectly plastic or bilinear approximations illustrated in Figure 1.4.

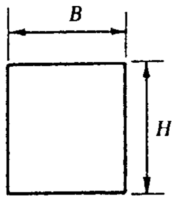
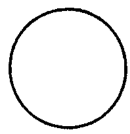
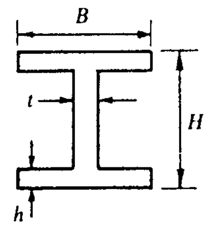
It is observed from equations (1.4) and (1.5) for a beam with a solid rectangular cross-section that

$$M_0 = 1.5M_y. \tag{1.6}$$

† The elastic behaviour of beams is considered in many textbooks, e.g., Venkatraman and Patel.[1.12]

‡ The stress field $\sigma_x = \pm\sigma_0$, $\sigma_y = \sigma_z = \sigma_{xy} = \sigma_{xz} = \sigma_{yz} = 0$, which accompanies the limit moment M_0, satisfies the equilibrium equations for a three-dimensional continuum, even though there is a discontinuity in σ_x at $z = 0$. It is evident from Figure 1.4 that the change in curvature κ, which is defined by equation (1.3), is positive when the limit moment M_0 is reached. This gives rise to a strain field ($\epsilon_x = z\kappa$, $\epsilon_y = \epsilon_z = -\nu z\kappa$, $\epsilon_{xy} = \epsilon_{xz} = \epsilon_{yz} = 0$) which satisfies the compatibility equations for a three-dimensional continuum.

Table 1.1. *Shape factors*

Beam cross-section			
$\frac{M_0}{M_y}$	1.50	1.70	$\frac{6+3\lambda}{6+2\lambda}$ $\lambda = \frac{Ht}{2Bh}$ $t \ll B, h \ll H$

The factor 1.5 is known as a shape factor and depends on the cross-sectional shape of a beam, as shown in Table 1.1.

The preceding discussion was developed for a beam which was subjected to a pure bending moment. In general, the loading on an actual beam would produce a much more complicated distribution of bending moments, which would be accompanied by lateral shear forces, as indicated by equations (1.1) and (1.2). However, it has been observed by Hodge[(1.2)] that the influence of these lateral shear forces on the magnitude of the plastic collapse moment of a cross-section may be disregarded for many structures which may be meaningfully called beams.† Thus, the limit moment at any location on an actual beam with a solid rectangular cross-section would be given by equation (1.5). In passing, it should be remarked that a kinematically admissible‡ collapse mechanism must form in order to develop the maximum strength of an actual beam. Consequently, the collapse load may well be larger than the shape factor times the load necessary to produce initial yielding of a beam (i.e., M_y).§

1.3 Plastic Collapse Theorems for Beams

1.3.1 Introduction

It was shown in the previous section that M_0 given by equation (1.6) is the plastic collapse or limit moment for a perfectly plastic beam with a solid rectangular cross-section when subjected to a pure bending moment. Clearly, the static load

† Transverse shear effects are sometimes important for the static loading of beams with open cross-sections, and design methods are available to cater for the combined influence of a transverse shear force and a bending moment on the plastic yielding of a beam cross-section.[(1.7)] However, transverse shear effects are potentially more important for dynamic loadings, as discussed in Chapter 6.

‡ A kinematically admissible collapse mechanism is a displacement field which satisfies the displacement boundary conditions, gives strains which satisfy the plastic incompressibility condition (constant volume) and allows the external loads to do positive work.

§ See footnote of equation (1.32) for a specific example.

carrying capacity of the cross-section is exhausted and collapses, as illustrated in Figure 1.4. However, what is the carrying capacity of a beam with an external load which produces a bending moment distribution which varies along the axis?

The limit theorems of plasticity have been developed in order to provide simple estimates for the static collapse loads of perfectly plastic beams subjected to any form of external static loading. The lower and upper bound theorems of plasticity, which uncouple the static (equilibrium) and kinematic (deformation) requirements of a theoretical solution, are introduced in the next two sections.

1.3.2 Lower Bound Theorem

1.3.2(a) Statement of Theorem

If any system of bending moments, which is in equilibrium with the applied loads and which nowhere violates the yield condition, can be found, then the beam will not collapse, or is at the point of collapse (incipient collapse).

1.3.2(b) Proof of Theorem

It is assumed that a set of external concentrated and distributed loads denoted by $F(x)$ just causes collapse (incipient collapse) of a beam. The associated collapse mechanism for the beam is characterised by a velocity profile $\dot{w}(x)$ and rotation rates $\dot{\theta}$ with $\dot{\theta}_i$ at i discrete locations (hinges). The bending moment distribution at collapse is $M(x)$ and M_i at the plastic hinges.

Now, the principle of virtual velocities† gives

$$\sum M_i \dot{\theta}_i = \int F \dot{w}\, \mathrm{d}x, \tag{1.7}$$

since M and F form an equilibrium set, while $\dot{\theta}$ and $\dot{w}$ are a kinematic set. All plastic hinge locations within the span of a beam and at the supports are included in the summation $\sum$, while the integral on the right-hand side of equation (1.7) extends over the entire beam.

The lower bound theorem of plasticity seeks to determine the multiplier λ^{l} so that the external load $\lambda^{\mathrm{l}} F(x)$ does not cause collapse and is safely supported by a beam.‡ The associated bending moment distribution $M^{\mathrm{s}}(x)$ is statically admissible when it satisfies the equilibrium equations (1.1) and (1.2) and nowhere exceeds the yield moment M_0 for the beam cross-section.

It is evident that M^{s} and $\lambda^{\mathrm{l}} F$ are in equilibrium, and therefore the principle of virtual velocities predicts

$$\sum M_i^{\mathrm{s}} \dot{\theta}_i = \int \lambda^{\mathrm{l}} F \dot{w}\, \mathrm{d}x, \tag{1.8}$$

which, when subtracted from equation (1.7), gives

$$(1 - \lambda^{\mathrm{l}}) \int F \dot{w}\, \mathrm{d}x = \sum (M_i - M_i^{\mathrm{s}}) \dot{\theta}_i. \tag{1.9}$$

† The principle of virtual velocities is discussed in Appendix 3.

‡ This is known as proportional loading because only a proportional combination of loads are considered.

The generalised stress (M) and generalised strain rate ($\dot{\theta}$) are defined to give a non-negative energy dissipation rate ($M\dot{\theta} \geq 0$), as observed in the footnote to equation (1.3).† Moreover, $|M^{\mathrm{s}}| \leq |M|$ throughout a beam according to the definition of a statically admissible bending moment field. Thus,

$$(M_i - M_i^{\mathrm{s}})\dot{\theta}_i \geq 0, \tag{1.10}$$

and, therefore, equation (1.9) predicts that

$$(1 - \lambda^{\mathrm{l}}) \int F\dot{w}\,\mathrm{d}x \geq 0,$$

or

$$\lambda^{\mathrm{l}} \leq 1, \tag{1.11}$$

since the external work rate $\int F\dot{w}\,\mathrm{d}x \geq 0$.

Equation (1.11) constitutes the proof of the lower bound theorem for beams which is stated in § 1.3.2(a).

1.3.3 Upper Bound Theorem

1.3.3(a) Statement of Theorem

If the work rate of a system of applied loads during any kinematically admissible collapse of a beam is equated to the corresponding internal energy dissipation rate, then that system of loads will cause collapse, or incipient collapse, of the beam.

1.3.3(b) Proof of Theorem

It is assumed that a beam collapses under a load $\lambda^{\mathrm{u}} F(x)$ with a bending moment field $M^{\mathrm{k}}(x)$ and an associated kinematically admissible velocity field $\dot{w}^{\mathrm{k}}(x)$, which has rotation rates $\dot{\theta}_j^{\mathrm{k}}$ at j discrete locations (plastic hinges). Thus, equating the external work rate to the internal energy dissipation during a kinematically admissible collapse gives

$$\sum M_j^{\mathrm{k}}\dot{\theta}_j^{\mathrm{k}} = \int \lambda^{\mathrm{u}} F\dot{w}^{\mathrm{k}}\,\mathrm{d}x, \qquad (1.12)^{\ddagger}$$

where M_j^{k} is the bending moment at the plastic hinges in the kinematically admissible collapse mechanism. Moreover,

$$\sum M_j\dot{\theta}_j^{\mathrm{k}} = \int F\dot{w}^{\mathrm{k}}\,\mathrm{d}x, \tag{1.13}$$

according to the principle of virtual velocities when using the equilibrium set (M, F) for the exact solution discussed in § 1.3.2(b). Subtracting equation (1.13) from equation (1.12) leads to

$$(\lambda^{\mathrm{u}} - 1) \int F\dot{w}^{\mathrm{k}}\,\mathrm{d}x = \sum (M_j^{\mathrm{k}} - M_j)\dot{\theta}_j^{\mathrm{k}}. \tag{1.14}$$

† If $M_i = M_0$, then $\dot{\theta}_i \geq 0$, while $\dot{\theta}_i \leq 0$, when $M_i = -M_0$. In both cases $M_i\dot{\theta}_i \geq 0$.
‡ Equation (1.12) is a definition for λ^{u}.

It is evident that $|M_j| \leq |M_j^{\mathrm{k}}|$, where $M_j^{\mathrm{k}} = \pm M_0$; and therefore $(M_j^{\mathrm{k}} - M_j)\dot{\theta}_j^{\mathrm{k}} \geq 0$, so that equation (1.14) requires

$$\lambda^{\mathrm{u}} \geq 1 \tag{1.15}$$

because $\int F\dot{w}^{\mathrm{k}}\mathrm{d}x \geq 0$. Inequality (1.15) constitutes the proof of the upper bound theorem for perfectly plastic beams which is stated in § 1.3.3(a).

1.3.4 Exact Static Collapse Load

The inequalities (1.11) and (1.15) may be written

$$\lambda^{\mathrm{l}} \leq 1 \leq \lambda^{\mathrm{u}}. \tag{1.16}$$

If

$$\lambda^{\mathrm{l}} = \lambda^{\mathrm{u}} = 1, \tag{1.17}$$

then a theoretical solution is simultaneously statically admissible (i.e., satisfies the requirements of the lower bound theorem) and kinematically admissible (i.e., satisfies the requirements of the upper bound theorem) and is, therefore, exact.

1.4 Static Plastic Collapse of a Cantilever

The limit theorems of plasticity, which were introduced in the previous section, are now used to obtain the static collapse load of the cantilever beam in Figure 1.5(a). The cantilever beam is made from a perfectly plastic material, is statically determinate, and has a linear bending distribution with a maximum value

$$M = -PL \tag{1.18}$$

at $x = 0$. Thus, the elastic stress distribution according to the elementary beam bending theory is $\sigma_x = zM/I$. This expression may be used to predict that a load

$$P_{\mathrm{E}} = 2\sigma_0 I/HL \tag{1.19}$$

could be supported in a wholly elastic manner when the beam cross-section is symmetric about the y-axis, where I is the second moment of area for the cross-section and σ_0 is the uniaxial plastic flow stress.

The bending moment distribution $P_{\mathrm{E}}(L - x)$, which is associated with the load P_{E}, satisfies the requirements of the lower bound theorem of plasticity in § 1.3.2. However, we observe from equation (1.18) that a higher lower bound is

$$P^{\mathrm{l}} = M_0/L, \tag{1.20}$$

since it gives $M = -M_0$ at $x = 0$ and produces a bending moment distribution which nowhere violates the yield condition for a beam with the rigid, perfectly plastic, or bilinear approximation characteristics in Figure 1.4. The extent of the plastic flow region, which is associated with P^{l}, is sketched in Figure 1.6 for a beam with a rectangular cross-section. In this case, the stress distribution across the beam cross-section at the support ($x = 0$) would be similar to that in Figure 1.2(d) but with the signs of the stresses reversed. By way of contrast, the plastic flow would be confined

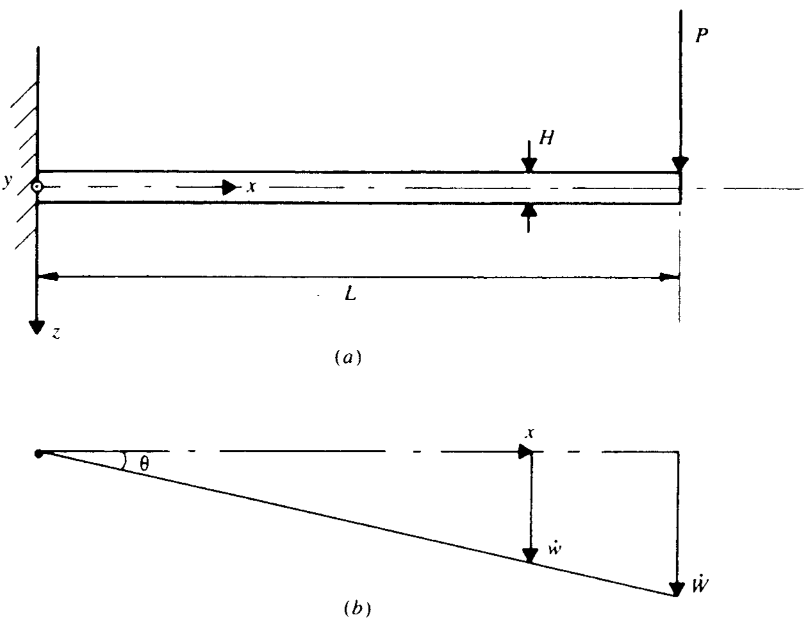


Figure 1.5. (*a*) Cantilever beam with an end load. (*b*) Transverse velocity profile for a cantilever beam with a plastic hinge at the support.

to the extreme elements ($z = \pm H/2$) located at $x = 0$ (i.e., locations A and B in Figure 1.6) when the beam is subjected to the load P_{E}.

The transverse velocity field, which is shown in Figure 1.5(*b*), may be employed to calculate an upper bound to the exact collapse load according to the method outlined in § 1.3.3. Therefore,

$$M_0\dot{\theta} = P^{\mathrm{u}} L\dot{\theta}$$

or

$$P^{\mathrm{u}} = M_0/L. \tag{1.21}$$

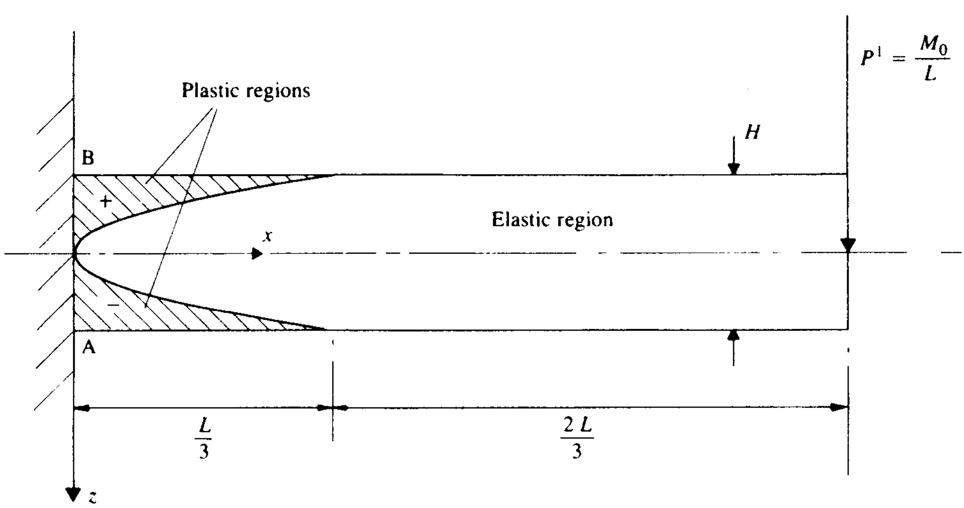


Figure 1.6. Elastic and plastic regions at collapse of the cantilever beam in Figure 1.5 with a rectangular cross-section.

Thus, the exact static collapse or limit load of the perfectly plastic cantilever beam, which is illustrated in Figure 1.5(a), is $P_c = M_0/L$, since both the lower and the upper bound calculations predict the same result. It is observed that the cantilever beam may support a load which is 50 percent larger than the maximum elastic value P_E given by equation (1.19) when the cross-section is rectangular ($I = BH^3/12$, $M_0 = BH^2\sigma_0/4$). In the case of a beam with a circular cross-section, then, according to Table 1.1, the plastic collapse load would be 1.70 times the initial yield value, which is predicted by an elementary linear elastic analysis.

The limit theorems of plasticity are valid for beams made from elastic, perfectly plastic or rigid, perfectly plastic materials. In other words, the exact static collapse load is identical for beams which are made from either material. In fact, it is evident from equation (1.5) in § 1.2 that the fully plastic bending moment M_0 is independent of the modulus of elasticity for the material. It is clear, therefore, that the limit theorems of plasticity bound the exact static plastic collapse load of a beam without any consideration of the complex elastic–plastic behaviour illustrated, for example, in Figure 1.6 for a cantilever beam with a rectangular cross-section.

1.5 Static Plastic Collapse of a Simply Supported Beam

The limit theorems of plasticity in § 1.3 are now used to obtain the limit load of the simply supported beam in Figure 1.7(*a*) which is made from a rigid, perfectly plastic material.

If a plastic hinge forms at the beam centre owing to the action of a uniformly distributed pressure p^u as shown in Figure 1.7(*b*), then an upper bound calculation (i.e., external work rate equals internal work rate) gives

$$2\,(p^u L)\left(L\dot{\theta}/2\right) = M_0 2\dot{\theta}$$

or

$$p^u = 2M_0/L^2. \tag{1.22}$$

The bending moment distribution in the region $0 \leq x \leq L$ of the beam in Figure 1.7(*a*) is

$$M = p\left(L^2 - x^2\right)/2, \qquad (1.23)^\dagger$$

which has the largest value

$$M = pL^2/2 \tag{1.24}$$

at the beam centre. Thus, the bending moment distribution (equation (1.23)) is statically admissible (i.e., $-M_0 \leq M \leq M_0$) for a pressure

$$p^l = 2M_0/L^2, \tag{1.25}$$

which when substituted into equation (1.23) gives

$$M/M_0 = 1 - (x/L)^2, \tag{1.26}$$

as shown in Figure 1.8.

† This expression may be obtained from a free body diagram for the portion of the beam of length $L - x$ or from the solution of the equilibrium equations (1.1) and (1.2) with $M = 0$ at $x = L$ and $Q = 0$ at $x = 0$.