Cambridge University Press
978-1-107-01034-5 - Gauge/Gravity Duality: Foundations and Applications
Martin Ammon & Johanna Erdmenger
Excerpt
More information

PART I

PREREQUISITES

Cambridge University Press
978-1-107-01034-5 - Gauge/Gravity Duality: Foundations and Applications
Martin Ammon & Johanna Erdmenger
Excerpt
More information

1 Elements of field theory

In this chapter we review some elements of quantum field theory which are essential in the study of gauge/gravity duality. A full development of quantum field theory is clearly beyond the scope of this book. We refer the reader to the many excellent textbooks on the subject, some of which are listed in the further reading section at the end of this chapter.

For simplicity, we will restrict ourselves to scalar fields in flat spacetime in the first part of the chapter and explain the most important concepts. We begin by introducing symmetries and conserved currents in the classical theory. A particularly important conserved quantity is the energy-momentum tensor which plays a central role in tests of the AdS/CFT correspondence. We discuss its derivation and its properties in detail. We then move on to the quantisation of field theories, beginning with the quantisation of the free scalar field. We review the definitions and concepts of generating functionals, of correlation functions and of the Feynman propagator. We then move on to interacting fields and discuss perturbation theory. Next we consider fermions as well as Abelian and non-Abelian gauge theories, both classically and in the quantised case. We discuss the energy-momentum tensor for classical gauge theories, as well as quantisation involving Faddeev–Popov ghost fields. An approximation of significance for gauge/gravity duality is the large N limit of non-Abelian gauge theories. Moreover, we discuss Ward identities and anomalies, which provide important examples of checks of the AdS/CFT correspondence later on.

1.1 Classical scalar field theory

Let us begin by introducing a real scalar field in flat d-dimensional Minkowski spacetime $\mathbb{R}^{d-1,1}$, with $d-1$ spatial directions. The points of the Minkowski spacetime are denoted by x with components x^μ, where μ runs from 0 to $d-1$. While $x^0 = ct$ is the time, x^i with $i = 1, \ldots, d-1$ are the spatial directions. In the following we set the speed of light c to one, $c = 1$, thus using the same units of measure for space and time. Sometimes it is also convenient to collect all the spatial components into a $(d-1)$-dimensional vector $\vec{x}$.

Minkowski spacetime is equipped with a metric. The infinitesimal length $\mathrm{d}s$ of a spacetime interval $\mathrm{d}x$ is given by

$$(\mathrm{d}s)^2 \equiv \mathrm{d}s^2 = -\left(\mathrm{d}x^0\right)^2 + \sum_{i=1}^{d-1} \left(\mathrm{d}x^i\right)^2 \equiv \eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu . \tag{1.1}$$

By definition, $\eta_{\mu\nu}$ is thus a diagonal matrix of the form

$$\eta_{\mu\nu} = \mathrm{diag}(-1, \underbrace{1, \ldots, 1}_{(d-1)\text{ times}}). \tag{1.2}$$

Using $\eta_{\mu\nu}$ or $\eta^{\mu\nu}$, which is the inverse of $\eta_{\mu\nu}$ satisfying $\eta^{\mu\nu}\eta_{\nu\sigma} = \delta^{\mu}_{\sigma}$, we may raise and lower the indices of x^μ, for example $x_\mu = \eta_{\mu\nu}x^\nu$. Equation (1.1) implies that $\mathrm{d}s^2$ can also be negative and therefore we do not have a metric in the strict mathematical sense. If $\mathrm{d}s^2 < 0$, the spacetime interval $\mathrm{d}x$ is *timelike*. For $\mathrm{d}s^2 = 0$ or $\mathrm{d}s^2 > 0$ the spacetime interval $\mathrm{d}x$ is *lightlike* or *spacelike*, respectively.

Let us now consider those transformations Λ of spacetime points $x \overset{\Lambda}{\mapsto} x'$ which leave $\mathrm{d}s^2$ invariant, i.e. for which

$$\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = \eta_{\mu\nu}\mathrm{d}x'^{\mu}\mathrm{d}x'^{\nu}. \tag{1.3}$$

It is easy to check that all transformations which satisfy equation (1.3) can be decomposed into translations of x by a constant vector a (with components a^μ), and into *Lorentz transformations* Λ given by the matrix components $\Lambda^{\mu}{}_{\nu}$ obeying

$$\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}\eta_{\mu\nu} = \eta_{\rho\sigma}. \tag{1.4}$$

For example, rotations in the spatial directions and boosts along a spatial direction are examples of Lorentz transformations. The Lorentz transformations Λ form a group, the Lorentz group $SO(d-1,1)$.

Both transformations, translations by a constant vector a and Lorentz transformations Λ, form a group, the *Poincaré* group $ISO(d-1,1)$, consisting of pairs (Λ, a) which act on spacetime as

$$x \mapsto x' = \Lambda x + a, \tag{1.5}$$

or in components $x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^\nu + a^\mu$. The group multiplication of two such operations (Λ_1, a_1) and (Λ_2, a_2) is given by

$$(\Lambda_1, a_1) \circ (\Lambda_2, a_2) = (\Lambda_1\Lambda_2, a_1 + \Lambda_1 a_2) \tag{1.6}$$

and is again in $ISO(d-1,1)$.

As an example of a field theory, we consider real scalar fields in d-dimensional Minkowski space. A real scalar field ϕ is a map which assigns a real number $\phi(x)$ to each spacetime point x. Under a Lorentz transformation $x \mapsto x' = \Lambda x$ the scalar field transforms as $\phi \mapsto \phi'$ where $\phi'(x') = \phi(x)$, or in terms of an active transformation $\phi'(x) = \phi(\Lambda^{-1}x)$. The dynamics of the scalar field is specified by an action functional $\mathcal{S}[\phi]$ which can be written as a spacetime integral of the *Lagrangian density* $\mathcal{L}(\phi, \partial_\mu\phi)$, or *Lagrangian* for short,

$$\mathcal{S}[\phi] = \int \mathrm{d}t\, \mathrm{d}^{d-1}\vec{x}\, \mathcal{L}(\phi, \partial_\mu\phi) \equiv \int \mathrm{d}^d x\, \mathcal{L}(\phi, \partial_\mu\phi). \tag{1.7}$$

The Lagrangian $\mathcal{L}$, and therefore also the action $\mathcal{S}$, depends on ϕ as well as its derivatives $\partial_\mu\phi$. For the partial derivative we use the shorthand notation $\partial_\mu \equiv \partial/\partial x^\mu$. We follow the usual approach to allow only first derivatives in the action functional and not second or higher derivatives of the scalar field. Moreover, we only consider local terms in the

Lagrangian, which means that terms of the form $\phi(x)\phi(x+a)$, where a is a spacetime vector, are not used. In order to formulate a scalar field theory which is invariant under Poincaré transformations, the action functional can only depend on ϕ, as well as on

$$-(\partial_t\phi(t,\vec{x}))^2 + (\nabla\phi(t,\vec{x}))^2 \equiv \eta^{\mu\nu}\partial_\mu\phi(x)\partial_\nu\phi(x). \tag{1.8}$$

The simplest example is the free scalar field theory given by the Lagrangian $\mathcal{L}_{\text{free}}$,

$$\begin{aligned}\mathcal{S}[\phi] = \int \mathrm{d}^d x\, \mathcal{L}_{\text{free}} &= -\frac{1}{2}\int \mathrm{d}^d x \left(-(\partial_t\phi(t,\vec{x}))^2 + (\nabla\phi(t,\vec{x}))^2 + m^2\phi(t,\vec{x})^2\right)\\ &= -\frac{1}{2}\int \mathrm{d}^d x \left(\eta^{\mu\nu}\partial_\mu\phi(x)\partial_\nu\phi(x) + m^2\phi(x)^2\right).\end{aligned} \tag{1.9}$$

The parameter m in the Lagrangian $\mathcal{L}_{\text{free}}$ is the mass of the scalar field ϕ. Varying the action $\mathcal{S}$ as given by (1.7) with respect to ϕ we obtain

$$\frac{\delta\mathcal{S}}{\delta\phi} = \frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right). \tag{1.10}$$

As usual, the classical equation of motion corresponding to the action $\mathcal{S}$ is determined by the principle of least action, $\delta\mathcal{S}/\delta\phi = 0$, and reads

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) = \frac{\partial\mathcal{L}}{\partial\phi}. \tag{1.11}$$

For the free field Lagrangian

$$\mathcal{L}_{\text{free}}(\phi,\partial_\mu\phi) = -\frac{1}{2}\eta^{\mu\nu}\partial_\mu\phi(x)\partial_\nu\phi(x) - \frac{1}{2}m^2\phi(x)^2 \tag{1.12}$$

the equation of motion (1.11) simplifies to

$$(\Box - m^2)\phi(x) = 0, \tag{1.13}$$

where $\Box = \partial^\mu\partial_\mu = -\partial_t^2 + \nabla^2$ is the *D'Alembert operator*. Equation (1.13) is known as the *Klein–Gordon equation*.

It is possible to add interaction terms to the free field Lagrangian $\mathcal{L}_{\text{free}}$, which are summarised in the *interaction Lagrangian* $\mathcal{L}_{\text{int}}$. Typically $\mathcal{L}_{\text{int}}$ is a polynomial of the field ϕ, for example

$$\mathcal{L}_{\text{int}}(\phi) = -\frac{g_n}{n!}\phi(x)^n, \tag{1.14}$$

where $n \geq 3, n \in \mathbb{N}$. The constant $g_n \in \mathbb{R}$ controls the strength of the interaction and is therefore referred to as the coupling constant.

Exercise 1.1.1 Show that the equations of motion (1.13) of a free scalar field are satisfied by

$$\phi(x) = \frac{1}{(2\pi)^{d-1}}\int \frac{\mathrm{d}^{d-1}\vec{k}}{2\omega_k}\left[a(\vec{k})e^{-ikx} + a^*(\vec{k})e^{ikx}\right]\Big|_{k^0=\omega_k}, \tag{1.15}$$

where $\omega_k = (\vec{k}\cdot\vec{k} + m^2)^{1/2}$ and $kx = -k^0x^0 + \vec{k}\vec{x}$.

Exercise 1.1.2 Derive the equations of motion for a scalar field with mass m and the interaction Lagrangian $\mathcal{L}_{\text{int}} = -\frac{g}{4!}\phi(x)^4$.

Exercise 1.1.3 Consider two non-interacting real scalar fields ϕ_1 and ϕ_2 with common mass m. Show that the Lagrangian can be written in terms of the *complex scalar field* $\phi = 1/\sqrt{2}\,(\phi_1 + i\phi_2)$ and its complex conjugate, $\phi^* = 1/\sqrt{2}\,(\phi_1 - i\phi_2)$ in the form

$$\mathcal{L}_{\text{free}}(\phi, \partial\phi) = -\partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi. \tag{1.16}$$

Derive the equations of motion for ϕ and ϕ^* assuming that ϕ and ϕ^* are independent fields. Are the equations of motion consistent with those for ϕ_1 and ϕ_2?

1.2 Symmetries and conserved currents

Symmetries are essential within field theory, and also play an essential role in the AdS/CFT correspondence. Let us first review the role of symmetries within classical field theory. One of the fundamental ingredients of theoretical physics is the intimate relation between continuous symmetries and conserved charges, as expressed in *Noether's theorem*. According to this theorem, a continuous symmetry gives rise to a conserved current which we now determine.

Let us assume that the action $\mathcal{S}[\phi]$ is invariant under the transformation

$$\phi(x) \;\mapsto\; \tilde{\phi}(x) \;=\; \phi(x) \;+\; \alpha\,\delta\phi(x), \tag{1.17}$$

where α denotes an arbitrary infinitesimal parameter associated with some deformation $\delta\phi$. The invariance of the action,

$$\mathcal{S}[\phi] = \mathcal{S}[\tilde{\phi}], \tag{1.18}$$

is ensured if the Lagrangian is also invariant under this deformation, up to a total derivative of some vector field X^μ,

$$\mathcal{L}\big(\tilde{\phi}, \partial_\mu\tilde{\phi}\big) \;=\; \mathcal{L}\big(\phi, \partial_\mu\phi\big) \;+\; \alpha\,\partial_\nu X^\nu \tag{1.19}$$

implying

$$\begin{aligned}
\alpha\,\partial_\nu X^\nu \;&\overset{!}{=}\; \mathcal{L}\big(\tilde{\phi}, \partial_\mu\tilde{\phi}\big) \;-\; \mathcal{L}\big(\phi, \partial_\mu\phi\big) = \mathcal{L}\big(\phi \;+\; \alpha\delta\phi, \partial_\mu\phi \;+\; \alpha\partial_\mu\delta\phi\big) \;-\; \mathcal{L}\big(\phi, \partial_\mu\phi\big)\\
&=\; \alpha\left\{\frac{\partial\mathcal{L}}{\partial\phi}\,\delta\phi \;+\; \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\partial_\mu\delta\phi\right\} + \mathcal{O}(\alpha^2)\\
&=\; \alpha\Bigg\{\underbrace{\left(\frac{\partial\mathcal{L}}{\partial\phi} \;-\; \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\right)}_{=0\text{ by }((1.11))}\delta\phi \;+\; \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\delta\phi\right)\Bigg\} + \mathcal{O}(\alpha^2)
\end{aligned} \tag{1.20}$$

or equivalently

$$0 \;\overset{!}{=}\; -\alpha\,\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\delta\phi\right) \;+\; \alpha\,\partial_\mu X^\mu \;=\; \alpha\,\partial_\mu\left(-\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\delta\phi \;+\; X^\mu\right). \tag{1.21}$$

This identifies a *conserved current* $\mathcal{J}^\mu$ associated with the *symmetry transformation* $\delta\phi$ of the field ϕ,

$$\mathcal{J}^\mu \;=\; -\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\,\delta\phi \;+\; X^\mu\,, \qquad \partial_\mu\mathcal{J}^\mu \;=\; 0. \tag{1.22}$$

Due to the conserved current $\mathcal{J}$, we may define an associated charge $\mathcal{Q}$, the *Noether charge*, by integration of the temporal component of $\mathcal{J}$, denoted by $\mathcal{J}^t$, over the spatial directions (given by $\mathbb{R}^{d-1}$) for a fixed value of time,

$$\mathcal{Q} = \int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}\, \mathcal{J}^t. \tag{1.23}$$

Exercise 1.2.1 By using Gauss' law, show that $\mathcal{Q}$ is time independent.

Let us discuss a few explicit examples of symmetries and associated Noether charges. Since the action $\mathcal{S}$ is invariant under Poincaré transformations by construction, we first construct the conserved current associated with spacetime translations of the form $x^\mu \mapsto x'^\mu = x^\mu + a^\mu$. Such transformations can be described alternatively as transformations of the field configuration

$$\phi(x) \mapsto \tilde{\phi}(x) = \phi(x-a) = \phi(x) - a^\mu \partial_\mu \phi(x) + \mathcal{O}(a^2), \tag{1.24}$$

under which the Lagrangian transforms as

$$\mathcal{L} \mapsto \tilde{\mathcal{L}} = \mathcal{L} - a^\nu \partial_\mu (\delta^\mu{}_\nu \mathcal{L}) + \mathcal{O}(a^2). \tag{1.25}$$

Let us now apply the Noether theorem with $\delta\phi = -a^\nu \partial_\nu \phi$ and $X^\mu = -\delta^\mu{}_\nu a^\nu \mathcal{L}$. We obtain a conserved current $\mathcal{J}^\mu = -a^\nu \Theta^\mu{}_\nu$, where

$$\Theta^\mu{}_\nu = -\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\nu \phi + \mathcal{L}\delta^\mu_\nu. \tag{1.26}$$

Note that $\Theta_{\mu\nu}$ is not manifestly symmetric by construction. However, if the Lagrangian takes the form $\mathcal{L} = \mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}}$, with $\mathcal{L}_{\text{free}}$ given by (1.12) and $\mathcal{L}_{\text{int}}$ independent of $\partial_\mu\phi$, then $\Theta_{\mu\nu}$ is given by

$$\Theta_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi + \eta_{\mu\nu}\mathcal{L} \tag{1.27}$$

and it turns out that Θ is symmetric, $\Theta_{\mu\nu} = \Theta_{\nu\mu}$. The associated conserved Noether charges are given by

$$H \equiv \int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}\, \mathcal{H} = \int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}\, \Theta^{tt} = \int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}\, (\Pi\, \partial_t \phi - \mathcal{L}) \tag{1.28}$$

for time translations as well as

$$P^i = \int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}\, \Theta^{ti} = -\int_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}\, \Pi\, \partial^i\phi \tag{1.29}$$

for space translations. H is the *Hamiltonian* and $\mathcal{H}$ the *Hamiltonian density*. Moreover, we have introduced the canonical momentum density $\Pi(t,\vec{x})$ conjugate to the field $\phi(t,\vec{x})$

$$\Pi(t,\vec{x}) = \frac{\partial\mathcal{L}}{\partial(\partial_t\phi(t,\vec{x}))}. \tag{1.30}$$

Furthermore, P_i is the *physical momentum* of the field ϕ. Equations (1.28) and (1.29) imply that the conserved current $\Theta_{\mu\nu}$ as given by equation (1.26) is the *energy-momentum tensor*.

Box 1.1 **Energy-momentum tensor in general relativity**

The energy-momentum tensor $T_{\mu\nu}$ is a key ingredient in general relativity since it determines the curvature of space by entering the Einstein equation. In section 2.2 we will introduce a second way of calculating $T_{\mu\nu}$ which by construction makes sure that $T_{\mu\nu}$ is symmetric in μ and ν.

Exercise 1.2.2 Show that for a free real scalar field ϕ with mass m, the Hamiltonian density is given by

$$\mathcal{H} = \frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2. \tag{1.31}$$

Instead of translations in space and time we can consider Lorentz transformations which are also a symmetry of the action. Under an infinitesimal Lorentz transformation, $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \omega^\mu{}_\nu$ with $\omega_{\mu\nu} = -\omega_{\nu\mu}$, the scalar field $\phi(x)$ transforms as $\phi(x^\mu) \mapsto \tilde{\phi}(x^\mu) = \phi(x^\mu - \omega^\mu{}_\rho x^\rho)$, i.e. with an x-dependent translation parameter $a^\mu = \omega^\mu{}_\rho x^\rho$. Using the same methods as above we conclude that

$$N^{\mu\nu\rho} = x^\nu \Theta^{\mu\rho} - x^\rho \Theta^{\mu\nu} \tag{1.32}$$

is conserved, i.e. $\partial_\mu N^{\mu\nu\rho} = 0$, and that the associated Noether charge is

$$M^{\nu\rho} = \int\limits_{\mathbb{R}^{d-1}} \mathrm{d}^{d-1}\vec{x}\; N^{t\nu\rho}(x). \tag{1.33}$$

Exercise 1.2.3 Use the conservation laws of $N^{\mu\nu\rho}$ and $\Theta^{\mu\nu}$ to show that any Poincaré invariant field theory has to have a symmetric energy-momentum tensor.

Note that the energy-momentum tensor $\Theta_{\mu\nu}$ as defined by (1.26) is not necessarily symmetric by construction. For the Lagrangian $\mathcal{L}_{\text{free}} + \mathcal{L}_{\text{int}}$ given by (1.12) and (1.14), $\Theta_{\mu\nu}$ is a symmetric tensor. Later we will see examples where the energy-momentum tensor as defined by (1.26) is not symmetric but is still conserved. However, note that we may add a term of the form $\partial_\lambda f^{\lambda\mu\nu}$ to $\Theta^{\mu\nu}$, with $f^{\lambda\mu\nu} = -f^{\mu\lambda\nu}$ antisymmetric in its first two indices, without spoiling the conservation laws. Due to the statement of exercise 1.2.3, there has to be a clever choice of $f^{\lambda\mu\nu}$ such that the tensor $T^{\mu\nu} = \Theta^{\mu\nu} + \partial_\lambda f^{\lambda\mu\nu}$ is still conserved but is also symmetric. $T^{\mu\nu}$ is the *Belinfante* or *canonical energy-momentum tensor*. Moreover, if we replace $\Theta^{\mu\nu}$ by $T^{\mu\nu}$ in (1.32) then $N^{\mu\nu\rho}$ is still conserved.

Exercise 1.2.4 For the massless free scalar field, we can refine the energy-momentum tensor even further to impose tracelessness in addition to conservation and index symmetry. In particular, show that the modified energy-momentum tensor given by

$$T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\partial_\rho\phi\partial^\rho\phi - \frac{d-2}{4(d-1)}\left(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box\right)\phi^2 \tag{1.34}$$

is symmetric, conserved and traceless, i.e. $T^\mu_\mu = \eta^{\mu\nu}T_{\mu\nu} = 0$, if we use the equations of motion of ϕ. Equation (1.34) is referred to as an *improved* energy-momentum tensor. In chapter 2 we will see that the last term in (1.34) is generated by coupling the scalar field to the Ricci scalar in a particular way referred to as *conformal*. The consequences of tracelessness of the energy-momentum tensor will be explored in chapter 3 when discussing conformal field theories.

In addition to spacetime symmetries, a further interesting type of symmetries is *internal symmetries*. For example, consider a complex scalar field as discussed in exercise 1.1.3. The Lagrangian (1.16) is invariant under the $U(1)$ transformation

$$\phi(x) \mapsto \phi(x)' = e^{i\alpha}\phi(x), \qquad \phi^*(x) \mapsto \phi^*(x)' = e^{-i\alpha}\phi^*(x). \tag{1.35}$$

This is an example of an internal symmetry. Since the parameter α is not spacetime dependent, the transformation is *global*.

Exercise 1.2.5 Determine the Noether currents associated with the global $U(1)$ transformation (1.35).

Exercise 1.2.6 Consider n free, real (or complex) fields ϕ^j with $j = 1, \ldots, n$ numbering the different fields. We assume the fields to be of the same mass, i.e. $m_j = m$. Determine the action and show that it is invariant under the transformation $\phi'^j(x) = R^j{}_k\phi^k(x)$ where $R^j{}_k$ are the components of a matrix R. In particular show that in the case of real scalar fields $R \in O(n)$, while for complex scalar fields $R \in O(2n) \supseteq U(n)$.[1]

1.3 Quantisation

Let us now quantise the classical scalar field theory using two different approaches: *canonical quantisation* and *path integral quantisation*. For canonical quantisation, the classical fields are promoted to operator valued quantum fields. On the other hand, the idea of path integral quantisation is to sum over all possible field configurations. Both approaches are discussed for free fields in 1.3.1 and 1.3.2.

In 1.3.3 we discuss interacting field theories. Particle scattering processes may be related to *correlation functions* of quantised fields which can be deduced from a *generating functional*. For quantisation of interacting fields, the approach that is best understood is *perturbation theory* which requires the couplings to be small. This implies that the majority of our current understanding of physical systems described by quantum field theories refers to weak coupling.

1.3.1 Canonical quantisation of free fields

We consider a massive real scalar field with equation of motion

$$(-\Box + m^2)\phi = 0. \tag{1.36}$$

We already discussed its solution in exercise 1.1.1 in terms of modes $a(\vec{k})$ and $a^*(\vec{k})$. The starting point of quantising the real scalar field is to promote these modes to operators $\hat{a}(\vec{k})$

[1] For generic interactions of complex scalar fields, the symmetry $O(2n)$ is typically broken down to $U(n)$ or even further.

and $\hat{a}^\dagger(\vec{k})$. The field $\phi(x)$ is then also operator valued and therefore denoted by $\hat{\phi}(x)$,

$$\hat{\phi}(x) = \frac{1}{(2\pi)^{d-1}} \int \frac{\mathrm{d}^{d-1}\vec{k}}{2\omega_k} \left[\hat{a}(\vec{k})e^{-ikx} + \hat{a}^\dagger(\vec{k})e^{ikx}\right]\Big|_{k^0=\omega_k}, \tag{1.37}$$

with $\omega_k = (\vec{k}\cdot\vec{k} + m^2)^{1/2}$. The operators $a(\vec{k})$ and $a^\dagger(\vec{k})$ satisfy the commutation relations

$$[\hat{a}(\vec{k}), \hat{a}^\dagger(\vec{k}')] = 2\omega_k(2\pi)^{d-1}\delta^{d-1}(\vec{k}-\vec{k}'), \quad [\hat{a}(\vec{k}), \hat{a}(\vec{k}')] = [\hat{a}^\dagger(\vec{k}), \hat{a}^\dagger(\vec{k}')] = 0. \tag{1.38}$$

Exercise 1.3.1 Using the commutation relations (1.38) show that $\hat{\phi}(t,\vec{x})$ and $\hat{\Pi}(t,\vec{x}) = \frac{\partial}{\partial t}\hat{\phi}(t,\vec{x})$ satisfy the equal-time commutation relations

$$\begin{aligned} \left[\hat{\phi}(t,\vec{x}), \hat{\Pi}(t,\vec{y})\right] &= i\delta^{d-1}(\vec{x}-\vec{y}), \\ \left[\hat{\phi}(t,\vec{x}), \hat{\phi}(t,\vec{y})\right] &= \left[\hat{\Pi}(t,\vec{x}), \hat{\Pi}(t,\vec{y})\right] = 0. \end{aligned} \tag{1.39}$$

Exercise 1.3.2 Show that the measure $\mathrm{d}^{d-1}\vec{k}/(2\omega_k)$ is invariant under Lorentz transformations by rewriting it in the form

$$\int \frac{\mathrm{d}^{d-1}\vec{k}}{2\omega_k} = \int \mathrm{d}^{d-1}\vec{k} \int \mathrm{d}k^0 \delta^d(k^2+m^2)\Theta(k^0), \tag{1.40}$$

where Θ is the step function defined by $\Theta(k^0) = 1$ for $k^0 > 0$ and $\Theta(k^0) = 0$ for $k^0 < 0$.

The commutation relations (1.38) are similar to those of a quantum harmonic oscillator. Therefore we interpret the operators $\hat{a}^\dagger(\vec{k})$ and $\hat{a}(\vec{k})$ as creation and annihilation operators of particles with momentum $\vec{k}$, respectively. The vacuum state $|0\rangle$ of the theory is then given by

$$\hat{a}(\vec{k})\,|0\rangle = 0. \tag{1.41}$$

We assume the normalisation $\langle 0|0\rangle = 1$. A single-particle state with momentum $\vec{k}$, denoted by $|\vec{k}\rangle$ can be created by acting on the vacuum state with the creation operator $\hat{a}^\dagger(\vec{k})$,

$$|\vec{k}\rangle = \hat{a}^\dagger(\vec{k})|0\rangle. \tag{1.42}$$

Multi-particle states $|\vec{k}_1, \vec{k}_2, \ldots\rangle$ can be similarly constructed by applying a product of creation operators $\hat{a}^\dagger(\vec{k}_1)\hat{a}^\dagger(\vec{k}_2)\ldots$ to the vacuum state $|0\rangle$.

1.3.2 Path integral quantisation of free fields

Within quantum mechanics, the path integral sums over all possible paths which start at some position q_i at time t_i and end at a position q_f at time t_f. In quantum field theory, this translates into summing over all field configurations ϕ in configuration space. The integration measure becomes formally

$$\mathcal{D}\phi \propto \prod_{t_\mathrm{i}\le t\le t_\mathrm{f}} \prod_{\vec{x}\in\mathbb{R}^{d-1}} \mathrm{d}\phi(t,\vec{x}). \tag{1.43}$$

The transition from an initial state $|\phi_\mathrm{i}, t_\mathrm{i}\rangle$ to a final state $|\phi_\mathrm{f}, t_\mathrm{f}\rangle$ where

$$\hat{\phi}(t_\mathrm{i},\vec{x})|\phi_\mathrm{i}, t_\mathrm{i}\rangle = \phi_\mathrm{i}(\vec{x})|\phi_\mathrm{i}, t_\mathrm{i}\rangle, \qquad \hat{\phi}(t_\mathrm{f},\vec{x})|\phi_\mathrm{f}, t_\mathrm{f}\rangle = \phi_\mathrm{f}(\vec{x})|\phi_\mathrm{f}, t_\mathrm{f}\rangle \tag{1.44}$$