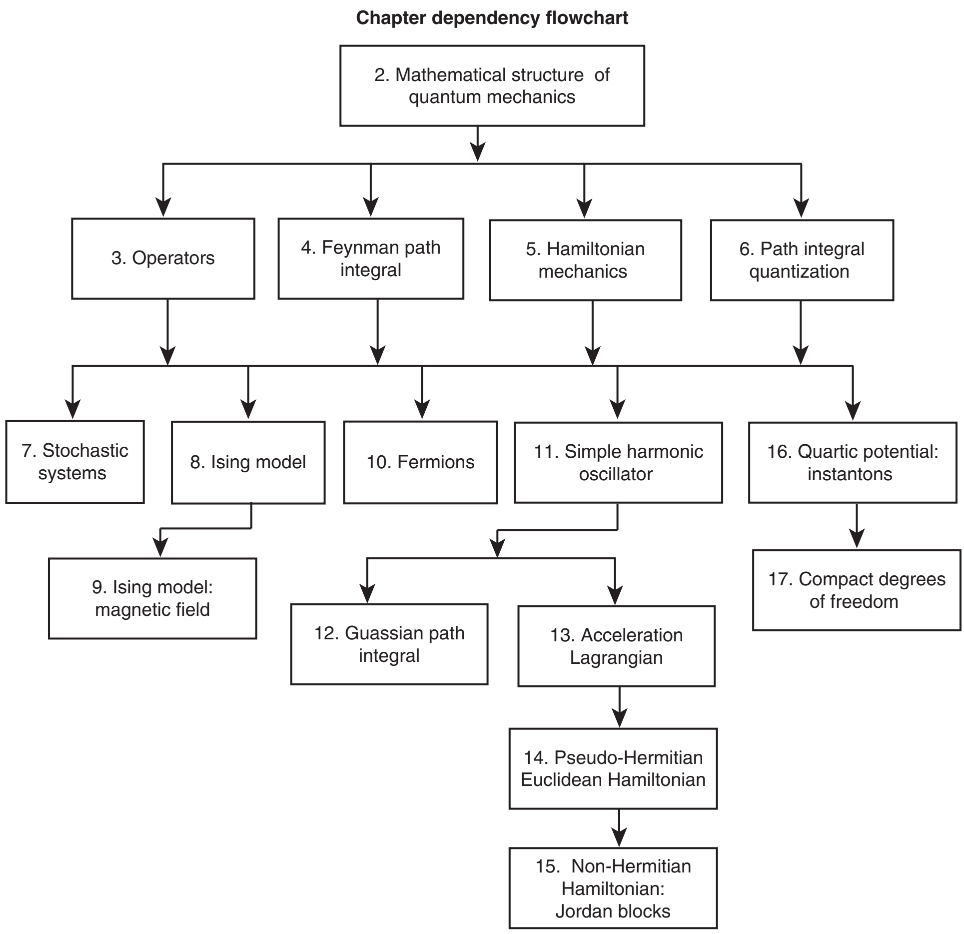
1

Synopsis

This book studies the mathematical aspect of path integrals and Hamiltonians – which emerge from the formulation of quantum mechanics. The theoretical framework of quantum mechanics provides the mathematical tools for studying both *quantum indeterminacy* and *classical randomness*. Many problems arising in quantum mechanics as well as in vastly different fields such as finance and economics can be addressed by the mathematics of quantum mechanics, or *quantum mathematics* in short. All the topics and subjects in the various chapters have been specifically chosen to illustrate the structure of quantum mathematics, and are not tied to any specific discipline, be it quantum mechanics or stochastic systems.

The book is divided into the following six parts, in accordance with the *Chapter dependency flowchart* given below.

- Part one addresses the *Fundamental principles* of path integrals and (Hamiltonian) operators and consists of five chapters. Chapter 2 is on the *Mathematical structure of quantum mechanics* and introduces the mathematical framework that emerges from the quantum principle. Chapters 3 to 6 discuss the mathematical pillars of quantum mathematics, starting from the Feynman path integral, summarizing Hamiltonian mechanics and introducing path integral quantization.
- Part two is on *Stochastic processes*. Stochastic systems are dissipative and are shown to be effectively modeled by the path integral. Chapter 7 is focused on the application of quantum mathematics to classical random systems and to stochastic processes.
- Part three discusses *Discrete degrees of freedom*. Chapters 8 and 9 discuss the simplest quantum mechanical degree of freedom, namely the double valued Ising spin. The Ising model is discussed in some detail as this model contains all the essential ideas that unfold later for more complex degrees of freedom. The general properties of path integrals and Hamiltonians are discussed in the context of the Ising spin. Chapter 10 on *Fermions* introduces a degree of freedom



that is essentially discrete – but is represented by fermionic variables that are distinct from real variables. The calculus of fermions, including the key structures of quantum mathematics such as the Hamiltonian, state space, and path integrals are discussed in some detail.

- Part four covers *Quadratic path integrals*. Chapter 11 is on the simple harmonic oscillator – one of the prime exemplars of quantum mechanics – and it is studied using both the Hamiltonian and path integral approach. In Chapter 12 different types of Gaussian path integrals are evaluated using techniques that are useful for analyzing and solving path integrals.
- Part five is on the *Acceleration action*. An action with an acceleration term is defined for Euclidean time and is shown to have a novel structure not present in usual quantum mechanics. In Chapter 13, the Lagrangian and path integral are

analyzed and shown to be equivalent to a constrained system. The Hamiltonian is obtained using the Dirac constraint method. In Chapter 14, the acceleration Hamiltonian is shown to be pseudo-Hermitian and its state space and propagator are derived. Chapter 15 examines a critical point of the acceleration action and the Hamiltonian is shown to be essentially non-Hermitian, being block diagonal and with each block being a Jordan block.

- Part six is on *Nonlinear path integrals*. Chapter 16 studies the nonlinear quartic Lagrangian to illustrate the qualitatively new features that nonlinear path integrals exhibit. The double well potential is studied in some detail as an exemplar of nonlinear path integrals that can be analyzed using the semi-classical expansion. And lastly, in Chapter 17 degrees of freedom are analyzed that take values in a compact manifold; these systems have a nonlinearity that arises from the nature of the degree of freedom itself – rather than from a nonlinear piece in the Lagrangian. Semi-classical expansions of the path integral about multiple classical solutions, classified by a winding number and path integrals on curved manifolds, are briefly touched upon.

Cambridge University Press
978-1-107-00979-0 - Path Integrals and Hamiltonians: Principles and Methods
Belal E. Baaquie
Excerpt
More information

Part one

Fundamental principles

Cambridge University Press
978-1-107-00979-0 - Path Integrals and Hamiltonians: Principles and Methods
Belal E. Baaquie
Excerpt
More information

2

The mathematical structure of quantum mechanics

An examination of the postulates of quantum mechanics reveals a number of fundamental mathematical constructs that form its theoretical underpinnings. Many of the results that are summarized in this Chapter will only become clear after reading the rest of the book and a re-reading may be in order.

The dynamical variables of classical mechanics are superseded by the quantum degree of freedom. An exhaustive and complete description of the indeterminate degree of freedom is given by its state function, which is an element of a state space that, in general, is an infinite-dimensional linear vector space. The properties of the indeterminate degree of freedom are extracted from its state vector by the linear action of operators representing experimentally observable quantities. Repeated applications of the operators on the state function yield the average value of the operator for the state [Baaquie (2013e)].

The conceptual framework of quantum mechanics is discussed in Section 2.1. The concepts of degree of freedom, state space and operators are briefly reviewed in Sections 2.3–2.5. Three distinct formulations of quantum mechanics emerge from the superstructure of quantum mechanics and these are briefly summarized in Sections 2.7–2.9.

2.1 The Copenhagen quantum postulate

The Copenhagen interpretation of quantum mechanics, pioneered by Niels Bohr and Werner Heisenberg, provides a conceptual framework for the interpretation of the mathematical constructs of quantum mechanics and is the standard interpretation that is followed by the majority of practicing physicists [Stapp (1963), Dirac (1999)].

The Copenhagen interpretation is not universally accepted by the physics community, with many alternative explanations being proposed for understanding quantum mechanics [Baaquie (2013e)]. Instead of entering this debate, this book is based on the Copenhagen interpretation, which can be summarized by the following postulates:

- The quantum entity consists of its degree of freedom $\mathcal{F}$ and its state vector $\psi(t, \mathcal{F})$, where t is a real number parameterizing time. The foundation of the quantum entity is its degree of freedom, which takes a range of values and constitutes a space $\mathcal{F}$. The quantum degree of freedom is completely described by the quantum state $\psi(t, \mathcal{F})$, a complex valued function of the degree of freedom that is an element of state space $\mathcal{V}(\mathcal{F})$.
- The quantum entity is an *inseparable pair*, namely, the degree of freedom and its state vector.
- All physically observable quantities are obtained by applying Hermitian operators $\mathcal{O}(\mathcal{F})$ on the state $\psi(t, \mathcal{F})$.
- Experimental observations collapse the quantum state and repeated observations yield $E_\psi[\mathcal{O}(\mathcal{F})]$, which is the expectation value of the operator $\mathcal{O}(\mathcal{F})$ for the state $\psi(t, \mathcal{F})$.
- The Schrödinger equation determines the time dependence of the state vector, namely of $\psi(t, \mathcal{F})$, but does not determine the process of measurement.

It needs to be emphasized that the state vector $\psi(t, \mathcal{F})$ provides only statistical information about the quantum entity; the result of any particular experiment is impossible to predict.[1]

The organization of the theoretical superstructure of quantum mechanics is shown in Figure 2.1.

The quantum state $\psi(t, \mathcal{F})$ is a complex number that describes the degree of freedom and is more fundamental than the observed probabilities, which are always real positive numbers. The scheme of assigning expectation values to operators, such as $E_\psi[\mathcal{O}(\mathcal{F})]$, leads to a generalization of classical probability to quantum probability and is discussed in detail in Baaquie (2013e).

To give a concrete realization of the Copenhagen quantum postulate, consider a quantum particle moving in one dimension; the degree of freedom is the real line, namely $\mathcal{F} = \Re = \{x | x \in (-\infty, +\infty)\}$ with state $\psi(t, \Re)$. Consider the position operator $\mathcal{O}(x)$;[2] a measurement projects the state to a point $x \in \Re$ and collapses the quantum state to yield, after repeated measurements

$$P(t, x) \equiv E_\psi[\mathcal{O}(x)] = |\psi(t, x)|^2, \quad P(t, x) > 0, \quad \int_{-\infty}^{+\infty} dx P(t, x) = 1. \quad (2.1)$$

[1] There are special quantum states called eigenstates for which one can exactly predict the outcome of some experiments. But for even this special case the degree of freedom is indeterminate and can never be directly observed.

[2] The position projection operator $\mathcal{O}(x) = |x\rangle\langle x|$; see Chapter 3.

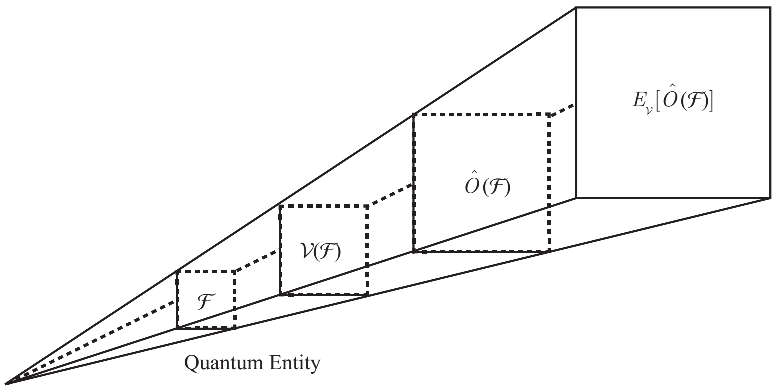


Figure 2.1 The theoretical superstructure of quantum mechanics; the quantum entity is constituted by the degree of freedom $\mathcal{F}$ and its state vector, which is an element of state space $\mathcal{V}(\mathcal{F})$; operators $\mathcal{O}(\mathcal{F})$ act on the state vector to extract information about the degree of freedom and lead to the final result $E_{\mathcal{V}}[\mathcal{O}(\mathcal{F})]$; only the final result, which is furthest from the quantum entity, is empirically observed.

Note from Eq. 2.1 that $P(t, x)$ obeys all the requirements to be interpreted as a probability distribution. A complete description of a quantum system requires specifying the probability $P(t, x)$ for all the possible projection operators of the quantum system. For a quantum particle in space, these are labeled by the different positions $x \in [-\infty, +\infty]$.

The position of the quantum particle is indeterminate and $P(t, x) = |\psi(t, x)|^2$ is the probability of the state vector collapsing at time t and at $\mathcal{O}(x)$ – the projection operator at position x. The moment that the state $\psi(t, \Re)$ is observed at *specific* projection operator $\mathcal{O}(x)$, the state $\psi(t, \Re)$ instantaneously becomes zero everywhere else. The transition from $\psi(t, \Re)$ to $|\psi(t, x)|^2$ is an expression of the collapse of the quantum state. It needs to be emphasized that no classical wave undergoes a collapse on being observed; the collapse of the state $\psi(t, \Re)$ is a purely quantum phenomenon.

The pioneers of quantum mechanics termed it as "wave mechanics" since the Newtonian description of the particle by its trajectory $x(t)$ was replaced by the state $\psi(t, \Re)$ that looked like a classical wave that is spread over (all of) space $\Re$. Hence the term "wave function" is used by many physicists for denoting $\psi(t, \Re)$.

The state $\psi(t, \mathcal{F})$ of a quantum particle is *not* a classical wave; rather, the only thing it has in common with a classical wave is that it is sometimes spread over space. However, there are quantum states that are *not spread over space*. For example, the up and down spin states of a quantum particle exist at a single point; such quantum states are described by a state that has no dependence on space and hence is *not* spread over space.

In the text, the terms *state, quantum state, state function*, or *state vector* are henceforth used for $\psi(t, \mathcal{F})$, as these are more precise terms than the term wave function.

The result given in Eq. 2.1 is an expression of the great discovery of quantum theory, namely, that *behind what is directly observed* – the outcome of experiments from which one can compute the probabilities $P(t, x) = |\psi(t, x)|^2$ – there lies an *unobservable world of the probability amplitude* that is fully described by the quantum state $\psi(t, \mathcal{F})$.

2.2 The superstructure of quantum mechanics

The description and dynamics of a quantum entity require an elaborate theoretical framework. The quantum entity is the foundation of the mathematical superstructure that consists of five main constructs:

- The quantum degree of freedom space $\mathcal{F}$.
- The quantum state vector $\psi(t, \mathcal{F})$, which is an element of the linear vector state space $\mathcal{V}(\mathcal{F})$.
- The time evolution of $\psi(t, \mathcal{F})$, given by the Schrödinger equation.
- Operators $\mathcal{O}(\mathcal{F})$ that act on the state space $\mathcal{V}(\mathcal{F})$.
- The process of measurement, with repeated observations yielding the expectation value of the operators, namely $E_\psi[\mathcal{O}(\mathcal{F})]$.

The five mathematical pillars of quantum mechanics are shown in Figure 2.2.

2.3 Degree of freedom space $\mathcal{F}$

Recall that in classical mechanics a system is described by dynamical variables, and its time dependence is given by Newton's equations of motion. In quantum mechanics, the description of a quantum entity starts with the generalization of the classical dynamical variables and, following Dirac (1999), is called the quantum *degree of freedom*.

Degree of freedom	State space	Dynamics	Operators	Observation
$\mathcal{F}$	$\mathcal{V}(\mathcal{F})$	$\frac{\partial\psi(t,\mathcal{F})}{\partial t}$	$O(\mathcal{F})$	$E_\psi[O(\mathcal{F})]$

Figure 2.2 The five mathematical pillars of quantum mechanics.