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LINEAR ALGEBRAIC GROUPS AND FINITE GROUPS OF LIE TYPE

Originating from a summer school taught by the authors, this concise treatment includes many of the main results in the area. An introductory chapter describes the fundamental results on linear algebraic groups, culminating in the classification of semisimple groups. The second chapter introduces more specialized topics in the subgroup structure of semisimple groups, and describes the classification of the maximal subgroups of the simple algebraic groups. The authors then systematically develop the subgroup structure of finite groups of Lie type as a consequence of the structural results on algebraic groups. This approach will help students to understand the relationship between these two classes of groups.

The book covers many topics that are central to the subject, but missing from existing textbooks. The authors provide numerous instructive exercises and examples for those who are learning the subject as well as more advanced topics for research students working in related areas.

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Linear Algebraic Groups and Finite Groups of Lie Type

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Contents

Preface

These notes grew out of a summer school on "Finite Groups and Related Geometrical Structures" held in Venice from September 5th to September 15th 2007. The aim of the course was to introduce an audience consisting mainly of PhD students and postdoctoral researchers working in finite group theory and neighboring areas to results on the subgroup structure of linear algebraic groups and the related finite groups of Lie type.

As will be seen in Part I, a linear algebraic group is an affine variety which is equipped with a group structure in such a way that the binary group operation and inversion are continuous maps. A connected (irreducible) linear algebraic group has a maximal solvable connected normal subgroup such that the quotient group is a central product of simple algebraic groups, a so-called semisimple algebraic group. Thus, one is led to the study of semisimple groups and connected solvable groups. A connected solvable linear algebraic group is the semidirect product of the normal subgroup consisting of its unipotent elements with an abelian (diagonalizable) subgroup (for example, think of the group of invertible upper triangular matrices). While one cannot expect to classify unipotent groups, remarkably enough this is possible for the semisimple quotient.

The structure theory of semisimple groups was developed in the middle of the last century and culminated in the classification of the semisimple linear algebraic groups defined over an algebraically closed field, a result essentially due to Chevalley, first made available via the *Séminaire sur la classification des groupes de Lie algébriques* at the Ecole Normale Supérieure in Paris, during the period 1956–1958 ([15]). Analogously to the work of Cartan and Killing on the classification of the complex semisimple Lie algebras, Chevalley showed that the semisimple groups are determined up to isomorphism by a set of combinatorial data, based principally upon a root system (as for the semisimple Lie algebras) and a dual root system. Moreover, the set of possible combinatorial data does not depend on the characteristic of the underlying field. Part I of this text is devoted to developing the tools necessary for describing this classification. We have followed the development in two very good texts ([32] and [66]) on linear algebraic groups, and we often refer to these books for the proofs which we have omitted. Our aim is to give the reader a feel for the group-theoretic ingredients of this classification, without going into the details of the underlying algebraic geometric foundations, and then to move on to the material of Parts II and III, which should perhaps be seen as the distinguishing feature of this text.

The 20- to 30-year period following the classification was a productive time in "semisimple" theory, during which many actors, notably Borel, Bruhat, Springer, Steinberg and Tits, played a role in the further study of these groups. The conjugacy classes, endomorphisms, representations, and subgroup structure were among the topics of consideration, with the principal aim of reducing their classification and the description of their structure to combinatorial data related to the root system and the Weyl group of the ambient group. Part II of this book treats some of these subjects. In particular, we describe the Tits BN-pair for a semisimple linear algebraic group and obtain a Levi decomposition for the associated parabolic subgroups; we discuss conjugacy classes of semisimple elements and their centralizers; we describe the parametrization of the irreducible representations of these groups and their automorphism groups. We leave the discussion of the general endomorphisms of simple algebraic groups to Part III, where these are used to construct the finite groups of Lie type.

In the last chapters of Part II, we turn to more recent developments in the theory of semisimple algebraic groups, where we describe the classification of their maximal positive-dimensional subgroups. These results can be seen as an extension of the fundamental work of Dynkin on the maximal subalgebras of the semisimple complex Lie algebras. It is the subject of several very long research articles by Liebeck, Seitz and others and was completed in 2004.

In the course of the classification of finite simple groups, attention turned to the analogues of algebraic groups over finite fields. These so-called finite groups of Lie type were eventually shown to comprise, together with the alternating groups, almost all the finite simple groups. Steinberg found a unified approach to constructing not only the well-known finite classical groups, but also their twisted "Steinberg variations", as well as further seemingly sporadic examples, the Suzuki and Ree groups, as fixed point subgroups of certain endomorphisms of simple linear algebraic groups defined over $\overline{\mathbb{F}_p}$. There does not yet seem to be a generally accepted terminology for such endomorphisms, and we call them "Steinberg endomorphisms" in this text.

Part III is devoted to the definition and study of these finite groups. We first classify the endomorphisms whose fixed point subgroups are finite, following the work of Steinberg, and hence are able to describe the full set of finite groups "of Lie type" thus obtained. The theorem of Lang–Steinberg then provides the necessary machinery for applying the results of Parts I and II to the study of these fixed point subgroups. For example, we give the proof of the existence of a BN-pair for these groups, which allows one to deduce a formula for their order. Furthermore, we study their Sylow subgroups and touch on some other aspects of the subgroup structure. Finally, we return to the question of the maximal subgroups, this time sketching a proof of Aschbacher's reduction theorem for the maximal subgroups of the finite classical groups and indicating how it has been applied (by Kleidman, Liebeck and others) and how it must still be applied if one hopes to determine the maximal subgroups of the finite classical groups. We conclude with a discussion of what is known about the maximal subgroups of the exceptional finite groups of Lie type, including work of Liebeck, Saxl, Seitz and others. We then come full-circle and sketch the proof of a result which enables one to lift certain embeddings of finite groups of Lie type to embeddings of algebraic groups, where one can apply the more complete information of Part II.

The course is not self-contained in several aspects. First, in order to keep the size manageable we assume the reader to be familiar with some basic notions of affine and projective algebraic varieties. In the development of the general theory of algebraic groups we include those proofs of a more group theoretical nature, or which just use the basic notions of connectedness and dimension, and refer to the standard texts for the others which require deeper methods from algebraic geometry, like properties of morphisms, tangent spaces, etc. Secondly, we do not explain the Steinberg presentation of semisimple algebraic groups, although some of its consequences are mentioned and needed in the text. Also, while we have included an appendix with a self-contained development of the basic theory of root systems and Weyl groups, as far as it is relevant for the development in the main text, we haven't repeated the proof of the classification of indecomposable root systems, which has already been laid out in many texts. In any case we give references to the results we need, and some statements form part of the exercises.

We hope that this text will be useful to doctoral students and researchers who are working in areas which rely upon a general knowledge of the groups

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Frontmatter
More information

of Lie type, without needing to understand every detail of the proof of the classification of semisimple groups. In particular, Parts II and III should give a good overview of much of what is known about the subgroup structure of these groups and to a lesser extent their conjugacy classes and representation theory. The numerous exercises are intended to supplement and illustrate the theory, and should help the book fulfill its objective of serving as the basis for a first-year graduate level course.

We began working on this project at the Mathematisches Forschungsinstitut at Oberwolfach even before the start of the summer school, and then continued at various places, including the EPFL (Lausanne), the Isaac Newton Institute at Cambridge, and the Banff International Research Station. We thank all these institutions for providing an inspiring atmosphere and enough fresh air, and for their hospitality. The second author would also like to acknowledge the support of the Swiss National Science Foundation through grants numbers PP002-68710 and 200021-122267.

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Finally, we would like to dedicate this volume to Maria Silvia Lucido, who died in an accident half a year after the summer school. We were both impressed by her enthusiasm, energy and joyfulness during our brief acquaintance.

Tables

Notation

We have tried to conform to standard notation whenever that exists. There are a few key notions for which different conventions exist in the literature. For us, a *reductive group* is not necessarily connected, while a *semisimple group* always is. A *simple algebraic group* is a non-trivial semisimple group with no proper positive dimensional normal subgroup. A root system which cannot be decomposed into an orthogonal union of subroot systems will be called *indecomposable* (sometimes the term irreducible is used in the literature). There does not seem to be an accepted standard notation for the various orthogonal groups. Here, the full isometry group of a non-degenerate quadratic form is denoted by GO, and its connected component of the identity by SO. In particular, in characteristic 2, SO is *not* the intersection of GO with the special linear group SL. We have chosen the name *Steinberg endomorphisms* for what some authors call (generalized) Frobenius maps, to acknowledge Steinberg's role in the study of these endomorphisms. Aschbacher gave a first subdivision of natural subgroups of classical groups over finite fields into classes which he called $\mathcal{C}_i$. Later, many of these classes were redefined by various authors. We follow the notation of Liebeck and Seitz in their paper [50] for these classes of subgroups. Finally, we write Z_n for the cyclic group of order n since C_n is already used for one of the root systems. We write $\mathbb{N} = \{1, 2, \ldots\}$ for the set of positive natural numbers and set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.