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Hydrodynamics of a one-component classical fluid

1.1 Thermodynamics of a one-component perfect fluid

In the strict sense of the word, hydrodynamics describes the dynamical behaviour of a fluid. But sometimes the hydrodynamical approach refers to phenomenological theories dealing with various types of condensed media, such as solids, liquid crystals, superconductors, magnetically ordered systems and so on. Two important and interconnected features characterise the hydrodynamical description.

- It refers to spatial and temporal scales much longer than any relevant microscopical scale of the medium under consideration.
- It does not need the microscopical theory for derivation of dynamical equations but uses as a starting point a set of conservation laws and thermodynamical and symmetry properties of the medium under consideration.

The latter feature gives us the possibility to study condensed matter without waiting for the moment when a closed self-consistent microscopical theory is developed. Sometimes it can be a long time to wait for such a moment. For example, one may recall the microscopical theory of fluid with strong interactions, or as the latest example the microscopical theory of high-T_c superconductivity. In fact, the cases when the hydrodynamical description can be derived rigorously from the 'first-principle' theory are more the exceptions rather than the rule. Such exceptions include, for example, weakly non-ideal gases and weak-coupling superconductors. Even if it is possible to derive the hydrodynamical description from the microscopical theory, the former as based on the most global properties (conservation laws and symmetry) is a reliable check of the microscopical theory. If hydrodynamics does not follow from a microscopical theory this is an alarming signal of potential problems with the microscopical theory.

Impressive evidence of the fruitfulness of the hydrodynamical (phenomenological) approach to condensed matter physics is provided by the volumes of Landau and Lifshitz's course addressing continuous media: *Electrodynamics of Continuous Media, Theory of Elasticity*, and *Fluid Mechanics* (Landau and Lifshitz, 1984, 1986, 1987). The hydrodynamical approach was very fruitful also for studying properties of rotating superfluids, as will be demonstrated in this book. The hydrodynamical description always deals with the

continuous medium even if the medium under consideration is a lattice (an atomic lattice in elasticity theory, for example). Indeed, the lattice constant is a microscopical scale which should be ignored in accordance with the nature of the hydrodynamical description. As a result, the hydrodynamical theories reduce to continuous field theories, and this similarity stimulated and is stimulating a useful exchange of ideas between condensed matter physics and field theory.

The basic idea of the hydrodynamical description is that any small volume of the fluid (small with respect to the hydrodynamical scales, but large with respect to any microscopical scale!) is in a state near to equilibrium described by thermodynamics, and we start from a discussion of the thermodynamics of the fluid.

The equilibrium state of any condensed medium, however complicated its microscopical properties, can be described by a few thermodynamical variables. Their number depends on the nature and symmetry of the condensed matter under consideration. The simplest are one-component fluids and gases, characterised by the highest symmetry and the minimal number of variables. The state of a resting one-component fluid (gas) in a very large volume V is completely defined by the mass and entropy density ρ and S. The energy density E_0 (the subscript 0 points out that a resting fluid is considered) as well as other thermodynamical parameters are functions of ρ and S. The surface effects are neglected, and the total mass $\mathcal{M} = \rho V$, the total entropy $\mathcal{S} = SV$, and the total energy $\mathcal{E}_0 = E_0 V$ are proportional to the volume V. For the energy density differential one has the Gibbs relation

$$dE_0 = \mu_0 d\rho + T dS, \tag{1.1}$$

where the chemical potential μ_0 and the temperature T are partial derivatives of the energy density:

$$\mu_0 = \frac{\partial E_0}{\partial \rho}, \qquad T = \frac{\partial E_0}{\partial S}. \tag{1.2}$$

Two contacting fluids are in equilibrium if their chemical potential and temperature are equal. It is often more convenient to use other pairs of variables rather than ρ and S. Correspondingly, a different thermodynamical potential other than the energy should be used to describe the equilibrium state (ensemble). This arises from the Legendre transformation. Choosing the mass density ρ and the temperature T as thermodynamical variables, the thermodynamical potential is the Helmholtz free energy for the canonical ensemble with density $F_0(\rho, T) = E_0 - TS$, which has the differential

$$dF_0 = d(E_0 - TS) = \mu_0 d\rho - S dT. \tag{1.3}$$

The great canonical ensemble is defined by μ_0 and T and the thermodynamical potential is the pressure

$$P = -E_0 + \mu_0 \rho + TS. \tag{1.4}$$

Equation (1.4) together with Eq. (1.1) gives the Gibbs–Duhem relation for the pressure differential:

$$dP = \rho d\mu_0 + SdT. \tag{1.5}$$

The pressure is directly connected with the mechanical work performed by an expanding fluid. Small work for a small adiabatic variation δV of the volume is

$$\delta A = -\frac{d\mathcal{E}_0}{dV}\delta V. \tag{1.6}$$

On adiabatic expansion the total mass and entropy are conserved, and the derivative $d\mathcal{E}_0/dV$ is calculated at fixed total mass ρV and total entropy SV. Therefore

$$\partial\rho/\partial V = -\rho/V, \qquad \partial S/\partial V = -S/V. \tag{1.7}$$

Then using Eq. (1.1) one obtains:

$$\begin{aligned}\frac{d\mathcal{E}_0}{dV} = \frac{d(E_0V)}{dV} &= E_0 + \frac{dE_0}{dV}V \\ &= E_0 + V\left(\mu_0\frac{d\rho}{dV} + T\frac{dS}{dV}\right) = E_0 - \mu_0\rho - TS.\end{aligned} \tag{1.8}$$

Comparing this with Eq. (1.4) we find that the pressure

$$P = -\frac{\partial\mathcal{E}_0}{\partial V} \tag{1.9}$$

determines the adiabatic work $\delta A = P\delta V$ indeed. If the fluid is moving with the centre-of-mass velocity $\boldsymbol{v}$, its momentum density or mass current is $\boldsymbol{j} = \rho\boldsymbol{v}$ while the fluid energy density is

$$E = E_0 + \rho v^2/2 = E_0 + j^2/2\rho. \tag{1.10}$$

The differential of the energy density is

$$dE = \mu_j d\rho + TdS + \boldsymbol{v}\cdot d\boldsymbol{j} \tag{1.11}$$

if the mass current $\boldsymbol{j} = \rho\boldsymbol{v}$ is used as a hydrodynamical variable. Here the chemical potential μ_j at fixed $\boldsymbol{j}$ is connected with the chemical potential μ_0 of the resting fluid by the relation

$$\mu_j = \mu_0 - v^2/2. \tag{1.12}$$

Calculating the pressure from the energy of the moving fluid, the total momentum $\boldsymbol{j}V$ of the fluid is kept fixed and

$$\partial\boldsymbol{j}/\partial V = -\boldsymbol{j}/V. \tag{1.13}$$

Then

$$P = -\partial\mathcal{E}/\partial V = -E + \mu_j\rho + TS + \boldsymbol{v}\cdot\boldsymbol{j} = -E_0 + \mu_0\rho + TS, \tag{1.14}$$

and the Gibbs–Duhem relation becomes

$$dP = \rho d\mu_j + SdT + \boldsymbol{j} \cdot d\boldsymbol{v} = \rho d\mu_0 + SdT. \tag{1.15}$$

So the pressure and the Gibbs–Duhem relation are not affected by the fluid motion.

Instead of $\boldsymbol{j}$ one may also choose the velocity $\boldsymbol{v}$ as a thermodynamic variable describing fluid motion. Then

$$dE = \mu d\rho + TdS + \boldsymbol{j} \cdot d\boldsymbol{v}, \tag{1.16}$$

and the chemical potential is

$$\mu = \mu_0 + \frac{v^2}{2}. \tag{1.17}$$

The pressure and the Gibbs–Duhem relation in this case also reduce to those in the coordinate frame where the fluid is at rest [Eqs. (1.14) and (1.17)].

1.2 Hydrodynamics of a one-component perfect fluid

In the hydrodynamical theory the thermodynamical variables (the chemical potential and the temperature) are not constant in general but vary smoothly in space and time. Instead of a tremendous number of microscopical variables (coordinates and velocities of atoms or their wave functions in the quantum mechanical description) one is dealing with continuous classical fields of a few thermodynamical variables describing the dynamical behaviour of the condensed medium. For a one-component perfect fluid, these are the mass density $\rho(\boldsymbol{R}, t)$, the entropy density $S(\boldsymbol{R}, t)$, and the mass current $\boldsymbol{j}(\boldsymbol{R}, t)$. Since they are slowly varying functions of the three-dimensional position vector $\boldsymbol{R}$ and the time t, for their local values one may use the thermodynamical relations given in the previous section. Hydrodynamical equations are derived from conservation laws. We follow the procedure described by Landau and Lifshitz (1987). A conservation law for some quantity means that the variation of its density in time at a given point in space is due only to flows from other parts of space. The mass conservation law gives the mass continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \boldsymbol{j} = 0. \tag{1.18}$$

There are conservation laws for the linear momentum,

$$\frac{\partial j_i}{\partial t} + \nabla_j \Pi_{ij} = 0, \tag{1.19}$$

and for the energy,

$$\frac{\partial E}{\partial t} + \nabla \cdot \boldsymbol{Q} = 0, \tag{1.20}$$

where Π_{ij} is the momentum flux tensor and $\boldsymbol{Q}$ is the energy flux. For a perfect (inviscid) fluid the total entropy is also conserved. This leads to the continuity equation for entropy

$$\frac{\partial S}{\partial t} + \nabla \cdot \boldsymbol{J}_S = 0. \tag{1.21}$$

Here $\boldsymbol{J}_S$ is the entropy flux.

All equations, the conservation laws included, must be invariant for any inertial coordinate frame (Galilean invariance). Suppose that there are two inertial coordinate systems $\boldsymbol{R}, t$ and $\boldsymbol{R}', t'$, the latter moving with the relative velocity $\boldsymbol{w}$:

$$\boldsymbol{R} = \boldsymbol{R}' + \boldsymbol{w}t', \qquad t = t', \qquad \boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{w}. \tag{1.22}$$

Then the time and space derivatives in the two frames are connected by the relations:

$$\frac{\partial}{\partial \boldsymbol{R}} = \frac{\partial}{\partial \boldsymbol{R}'}, \qquad \frac{\partial}{\partial t} = \frac{\partial}{\partial t'} - \left(\boldsymbol{w} \cdot \frac{\partial}{\partial \boldsymbol{R}'}\right). \tag{1.23}$$

Transforming the conservation laws (1.18)–(1.21) to the coordinate frame $(\boldsymbol{R}', t')$ with the help of Eqs. (1.22) and (1.23), we see that the equations are invariant (do not change their form) if the flows are transformed as

$$\boldsymbol{j} = \boldsymbol{j}' + \rho \boldsymbol{w}, \tag{1.24}$$

$$\boldsymbol{J}_S = \boldsymbol{J}'_S + S\boldsymbol{w}, \tag{1.25}$$

$$\Pi_{ij} = \Pi'_{ij} + j'_i w_j + w_i j'_j + \rho w_i w_j, \tag{1.26}$$

$$Q_i = Q'_i + E' w_i + w_j(\Pi'_{ji} + j'_j w_i) + \frac{w^2}{2}(j'_i + \rho w_i). \tag{1.27}$$

For the one-component isotropic fluid the only vector at our disposal is the velocity $\boldsymbol{v}$ (or the current $\boldsymbol{j}$ parallel to $\boldsymbol{v}$) and there are no tensors besides δ_{ij} and $v_i v_j$. One can check that the system of equations is Galilean invariant and all conservation laws are satisfied if the fluxes are given by the following expressions (Landau and Lifshitz, 1987):

$$\Pi_{ij} = P\delta_{ij} + \rho v_i v_j, \tag{1.28}$$

$$\boldsymbol{J}_S = S\boldsymbol{v}, \tag{1.29}$$

$$\boldsymbol{Q} = [\rho(\mu_j + v^2) + TS]\boldsymbol{v} = \left[\rho\left(\mu_0 + \frac{v^2}{2}\right) + TS\right]\boldsymbol{v}. \tag{1.30}$$

The Euler equation follows directly from the mass continuity equation and the momentum conservation law:

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\nabla)\boldsymbol{v} = -\frac{\nabla P}{\rho}. \tag{1.31}$$

The vector identity

$$(\boldsymbol{v}\nabla)\boldsymbol{v} = \nabla\frac{v^2}{2} + [[\nabla \times \boldsymbol{v}] \times \boldsymbol{v}] \tag{1.32}$$

transforms the Euler equation into another form:

$$\frac{\partial \boldsymbol{v}}{\partial t} + [\tilde{\boldsymbol{\omega}} \times \boldsymbol{v}] = -\frac{\boldsymbol{\nabla} P}{\rho} - \boldsymbol{\nabla}\frac{v^2}{2}. \tag{1.33}$$

Here $\tilde{\boldsymbol{\omega}} = [\boldsymbol{\nabla} \times \boldsymbol{v}]$ is the vorticity of the velocity. Later on in this chapter we neglect the temperature variation or consider the case $T = 0$. Then using the Gibbs–Duhem relation (1.5) at zero temperature, the Euler equation becomes

$$\frac{\partial \boldsymbol{v}}{\partial t} + [\tilde{\boldsymbol{\omega}} \times \boldsymbol{v}] = -\boldsymbol{\nabla}\left(\mu_0 + \frac{v^2}{2}\right). \tag{1.34}$$

On the right-hand side one sees the chemical potential μ [see Eq. (1.17)] for the fluid moving with velocity $\boldsymbol{v}$.

Let us discuss now the angular momentum (moment) conservation law. For an isotropic fluid, which consists of particles without internal degrees of freedom, i.e., without intrinsic angular momentum, the moment density $\boldsymbol{M}$ is not an independent variable, but is directly determined by the linear momentum density $\boldsymbol{j}$:

$$\boldsymbol{M} = [\boldsymbol{j} \times \boldsymbol{R}]. \tag{1.35}$$

Then

$$\frac{\partial M_i}{\partial t} = \epsilon_{ijk}\frac{\partial j_j}{\partial t}R_k = -\epsilon_{ijk}\nabla_n \Pi_{jn} R_k = -\nabla_n(\epsilon_{ijk}\Pi_{jn}R_k) + \epsilon_{ijk}\Pi_{jk}. \tag{1.36}$$

For any symmetric momentum flux tensor Π_{ij} this equation takes the form of the conservation law,

$$\frac{\partial M_i}{\partial t} + \nabla_n G_{in} = 0 \tag{1.37}$$

with the angular momentum flux tensor

$$G_{in} = \epsilon_{ijk}\Pi_{jn}R_k. \tag{1.38}$$

For an isotropic fluid the tensor Π_{ij} is symmetric indeed, as seen from Eq. (1.28).

Usually when deriving hydrodynamics one starts from the Euler equation and checks the momentum conservation law afterwards. We have shown here the opposite direction since it will be helpful for understanding two-fluid hydrodynamics. But it is useful to reproduce here the usual derivation also. One starts from Newton's second law for the unit volume of the fluid, which is

$$\rho\frac{d\boldsymbol{v}}{dt} = \boldsymbol{f}. \tag{1.39}$$

The force $\boldsymbol{f}$ per unit volume (the force density) is defined by the stress tensor,

$$f_i = -\nabla_j \sigma_{ij}, \tag{1.40}$$

which is a scalar for the isotropic fluid:

$$\sigma_{ij} = P\delta_{ij}. \tag{1.41}$$

The time derivative $d\boldsymbol{v}/dt$ in Eq. (1.39) is a substantial derivative in the Lagrange description of fluid dynamics (Batchelor, 1970). The velocity is considered as a Lagrange variable: taking the Lagrange (substantial) derivative one looks for a difference of two velocities of *the same fluid particle* in two instants of time at which, because of its motion, the particle is located in *different points in space*. On the other hand, the partial time derivative $\partial\boldsymbol{v}/\partial t$ in the Euler equation (1.31) is a derivative in the Euler description: taking the time Euler derivative one compares velocities of *two different particles*, which at two instants of time were located at *the same point of space*. The two derivatives are connected by the relation

$$\frac{d\boldsymbol{v}}{dt} = \frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v}. \tag{1.42}$$

As a result Eqs. (1.39)–(1.42) give the Euler equation Eq. (1.31).

1.3 Motion of a cylinder in an incompressible perfect fluid: backflow

We address an elementary problem of classical hydrodynamics considered in numerous textbooks: motion of a cylinder immersed in a perfect incompressible fluid. The problem is important for the analysis of vortex dynamics later in the book. Classic hydrodynamics tells us that a moving cylinder of radius R_0 induces a dipole velocity field around it (backflow):

$$\boldsymbol{V}_{bf}(\boldsymbol{r}) = \nabla\Phi_{bf} = -R_0^2\left[\frac{\boldsymbol{v}_{bf}}{r^2} - \frac{2(\boldsymbol{v}_{bf}\cdot\boldsymbol{r})\boldsymbol{r}}{r^4}\right]. \tag{1.43}$$

Here $\Phi_{bf} = -R_0^2\boldsymbol{v}_{bf}\cdot\boldsymbol{r}/r^2$ is the scalar velocity potential, the velocity $\boldsymbol{v}_{bf}$ is a constant which determines the backflow intensity, and $\boldsymbol{r}$ is a two-dimensional position vector normal to the cylinder axis (the axis z) with an origin at this axis. The velocity field of the backflow is shown in Fig. 1.1a.

The presence of backflow makes it possible to satisfy a natural condition that in the coordinate frame with velocity $\boldsymbol{v}_L$ of the cylinder, a current normal to the cylinder boundary must vanish. Suppose that far from the cylinder the fluid moves with velocity $\boldsymbol{v}_\infty$. Then the total fluid velocity field around the cylinder is $\boldsymbol{v}_\infty + \boldsymbol{V}_{bf}(\boldsymbol{r})$, and the radial current $\rho[\boldsymbol{v}_\infty + \boldsymbol{V}_{bf}(\boldsymbol{r}) - \boldsymbol{v}_L]\cdot\boldsymbol{r}/r$ vanishes at $r = R_0$ if $\boldsymbol{v}_{bf} = \boldsymbol{v}_L - \boldsymbol{v}_\infty$. The kinetic energy of the backflow in the coordinate frame moving with fluid velocity $\boldsymbol{v}$ is given by

$$\mu_a\frac{(\boldsymbol{v}_L-\boldsymbol{v}_\infty)^2}{2} = \frac{\rho R_0^4}{2}\int\limits_{r>R_0} d\boldsymbol{r}^2\left|\nabla\left[\frac{(\boldsymbol{v}_L-\boldsymbol{v}_\infty)\cdot\boldsymbol{r}}{r^2}\right]\right|^2 = \pi R_0^2\rho\frac{(\boldsymbol{v}_L-\boldsymbol{v}_\infty)^2}{2}. \tag{1.44}$$

This determines the mass $\mu_a = \pi R_0^2\rho$ per unit length, which is equal to the mass per unit length of the fluid inside a cylinder of radius R_0. It is called the *associated mass* since this mass must be added to the mass of the cylinder itself when determining the kinetic energy of a moving cylinder.

The associated mass must be connected with an additional momentum of the fluid dragged by the moving object. The calculation of this momentum has a subtlety which is

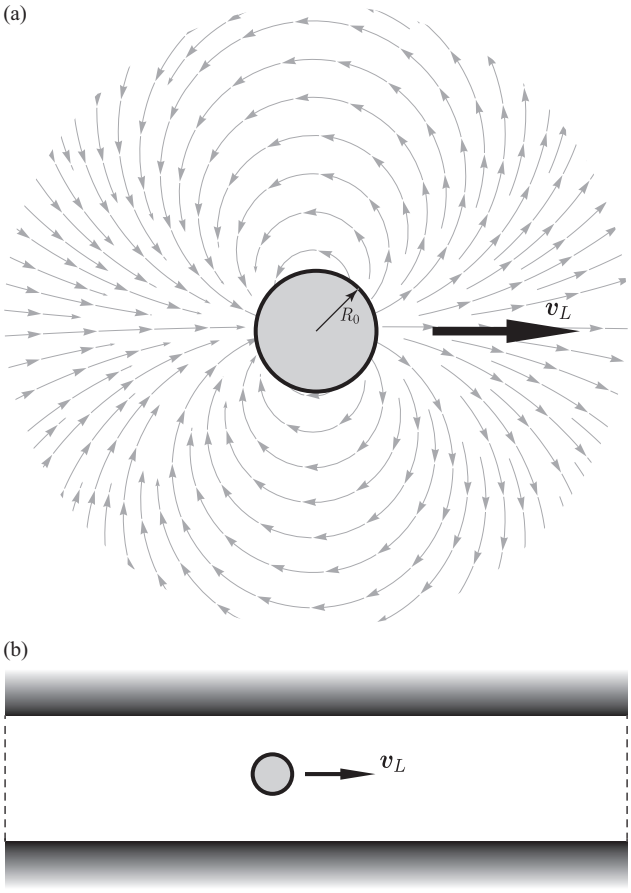

Figure 1.1 Cylinder of radius R_0 moving through a resting fluid with the velocity $\boldsymbol{v}_L$. (a) Velocity field of the backflow around the cylinder. (b) The cylinder moves in a straight channel with periodic boundary conditions at its ends.

well known in classical hydrodynamics. The momentum of the perfect fluid with potential velocity field $\boldsymbol{v}(\boldsymbol{R}) = \nabla\Phi(\boldsymbol{R})$ is determined by an integral, which after integrating by parts reduces to a surface integral over a surface S confining the fluid:

$$\boldsymbol{P} = \rho \int \boldsymbol{v}(\boldsymbol{R})\, d\boldsymbol{R} = \rho \int \nabla\Phi(\boldsymbol{R})\, d\boldsymbol{R} = \rho \int_S \Phi d\boldsymbol{S}. \tag{1.45}$$

Here $d\boldsymbol{S}$ is a vector with magnitude equal to the element dS of the surface area and directed normally to the surface outside the bulk occupied by the fluid. In the case of backflow, the surface S consists of the surface of the cylinder of radius R_0 and the surface S_∞ confining

the fluid at large distance from the cylinder. The momentum per unit length of the cylinder is

$$\boldsymbol{P} = \rho \int \nabla \Phi_{bf}(\boldsymbol{r})\, d\boldsymbol{r} = \rho \left(\int\limits_{r=R_0} \Phi_{bf} d\boldsymbol{S} + \int\limits_{S_\infty} \Phi_{bf} d\boldsymbol{S} \right). \tag{1.46}$$

This expression cannot define the momentum as long as the integral over S_∞ is undefined. Classical hydrodynamics (Lamb, 1997) tells us that the integral over S_∞ must simply be deleted from the expression. The remaining integral over the surface of the cylinder is called the Kelvin impulse:

$$\boldsymbol{P}_K = \rho \int\limits_{r=R_0} \Phi_{bf} d\boldsymbol{S} = \mu_a \boldsymbol{v}_{bf}. \tag{1.47}$$

In classical hydrodynamics the Kelvin impulse is justified by considering the momentum transferred to the object when making it move from rest (see Lamb, 1997, Sec. 119). Forces which make the object move from the state of rest are applied locally and do not affect the velocity and the velocity potential in the fluid far away from the object. This justifies ignoring the integral over S_∞ in the expression (1.46) for the momentum. The derivation in fact assumes that there is finite compressibility of the fluid and the momentum transferred to the fluid locally is not distributed over the whole fluid instantly.

In superfluid hydrodynamics one can justify using the Kelvin impulse in a simpler way (Sonin, 1973). In superfluids the velocity potential $\Phi = \hbar\theta/m$ is determined by the quantum mechanical phase θ of the condensate wave function (see Section 1.15). Let us consider a cylinder moving in a straight channel with periodic boundary conditions at its ends (Fig. 1.1b). The surface S_∞ contains lateral walls of the channel and cross-section planes at channel ends. Since there is no flow normal to the lateral walls, only the channel ends contribute to the momentum component along the channel. The periodic boundary condition requires that the phase difference at the channel ends is an integer number of 2π. In the absence of an essential transport current along the channel, the phase difference must vanish. Then the whole contribution of the surface S_∞ to the momentum also vanishes. This example of rather simple geometry illustrates the general rule that local perturbations of the velocity field cannot change the phase at infinity. On the other hand, motion of an object through an incompressible perfect fluid inevitably leads to a very small but still finite velocity at infinity. The finite momentum of the fluid dragged by the moving object means that that the whole fluid moves with average velocity $\boldsymbol{P}/\rho V$ inversely proportional to the volume V. The fluid moves with this velocity at the entrance to and the exit from the channel shown in Fig. 1.1b. This tiny velocity integrated over the whole volume gives a contribution of the same order as the momentum $\boldsymbol{P}$. According to Fig. 1.1a, the fluid in the backflow area around the cylinder moves in the direction opposite to the direction of the cylinder velocity $\boldsymbol{v}_L$. Nevertheless, at large distance from the cylinder the fluid moves with small average velocity in the same direction as the cylinder.

Although external forces are necessary in order to make an object move through a perfect fluid, no force is needed to support steady translational motion of the object. The latter moves without any resistance. In classical hydrodynamics this was called *d'Alembert's paradox* (Landau and Lifshitz, 1987), since it was in evident contradiction to real observations. Resolution of this paradox in classical hydrodynamics is quite evident: in a real fluid one cannot neglect viscosity (see Section 1.8 addressing motion of a cylinder through a viscous fluid). But for superfluids, d'Alembert's paradox ceases to be a paradox. It is a real observed phenomenon: at low velocities objects move through a superfluid without dissipation and without generation of vorticity. So in a sense d'Alembert's paradox is a precursor of the phenomenon of superfluidity.

1.4 Motion of a cylinder in an incompressible perfect fluid: Magnus force

A transverse force normal to the velocity of a moving cylinder appears even in a perfect fluid if there is a circular flow around the cylinder. The velocity field of the circular flow is

$$\boldsymbol{v}_v = \frac{[\boldsymbol{\kappa} \times \boldsymbol{r}]}{2\pi r^2}, \tag{1.48}$$

where the vector $\boldsymbol{\kappa}$ is parallel to the z axis. Its magnitude is the circulation of the velocity given by the linear integral $\kappa = \oint \boldsymbol{v}_v \cdot d\boldsymbol{l}$ over any closed path around the cylinder. In superfluids the circulation can only have special quantised values, whereas in classical fluids κ is arbitrary. In addition to the circular flow around the cylinder, there is a fluid current past the cylinder with constant velocity $\boldsymbol{v}_\infty$ (Fig. 1.2a). Then the net velocity field is

$$\boldsymbol{v}(\boldsymbol{r}) = \boldsymbol{v}_v(\boldsymbol{r}) + \boldsymbol{v}_\infty. \tag{1.49}$$

Streamlines of this velocity field are shown in Fig. 1.2b. Here we ignore the backflow around the moving cylinder, which has no contribution to the transverse force. The cylinder moves with a constant velocity $\boldsymbol{v}_L$, and one should replace the position vector $\boldsymbol{r}$ in the coordinate frame moving together with the cylinder by $\boldsymbol{r} - \boldsymbol{v}_L t$. Then the time derivative of the velocity is

$$\frac{\partial \boldsymbol{v}}{\partial t} = -(\boldsymbol{v}_L \cdot \nabla)\boldsymbol{v}. \tag{1.50}$$

In the coordinate frame moving with the cylinder, the velocity field does not vary in time, and the Euler equation (1.33) yields the Bernoulli law, which determines the pressure variation around the moving cylinder:

$$P = P_0 - \frac{\rho[\boldsymbol{v}(\boldsymbol{r}) - \boldsymbol{v}_L]^2}{2} = P_0' - \frac{\rho v_v(\boldsymbol{r})^2}{2} - \rho \boldsymbol{v}_v(\boldsymbol{r}) \cdot (\boldsymbol{v}_\infty - \boldsymbol{v}_L). \tag{1.51}$$

Here P_0 and $P_0' = P_0 - \frac{1}{2}\rho(\boldsymbol{v}_\infty - \boldsymbol{v}_L)^2$ are constants, which are of no importance for the further derivation. Figure 1.2b shows that due to superposition of two fluid motions given by Eq. (1.49), the velocity above the cylinder is higher than that below the cylinder. According to the Bernoulli law, the pressure is higher in the area where the velocity is