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Excerpt
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1 Quantized electromagnetic field and coherent state representations

The foundations of the quantum theory of radiation were laid by the work of Planck, Einstein, Dirac, Bose, Wigner, and many others. Historically Planck's [1] work on black body radiation is the foundation of any work on the quantum theory of radiation. Einstein's [2] work on the photoelectric effect established the particle nature of the radiation field. These particles were named as photons by Lewis [3]. Einstein [4] also introduced the A and B coefficients to describe the interaction of radiation and matter. He characterized stimulated emission using the B coefficient. Using thermodynamic arguments, he could also extract the A coefficient describing spontaneous emission which is at the heart of the origin of all spectral lines. This was quite a remarkable achievement. Dirac [5] implemented the quantization of the electromagnetic field and showed how Einstein's A coefficient emerges naturally from the quantization of the radiation field. It should be remembered that stimulated emission is the key to the working of any laser system. Following Dirac's quantization of the radiation field, Weisskopf and Wigner [6] were able explain in a very fundamental way the decay of the excited states of a system and hence derive the remarkable law of exponential decay. Bose [7] discovered a quantitative explanation for Planck's law. He introduced a new way of counting statistics relevant to quantum particles with zero mass. This was the beginning of quantum statistics. Bose's work was followed by Einstein [8] who produced a counting statistics for particles with finite mass (now known as Bosons). Fürth [9] studied the fluctuations in the energy distribution of black body radiation and thus shed light on the wave–particle duality of light. Einstein found that the fluctuations had two types of contributions – one can be interpreted in terms of the particle characteristics and the other in terms of the wave characteristics. Dirac also investigated the question of a proper phase operator for a quantized radiation field. This is because in classical physics the phase is extensively used to characterize coherent fields. He introduced one but also worried about its unitary nature. The question posed by Dirac on the phase operator became a subject of intense activity during the 1990s [10], but it is still not fully resolved. In this chapter we will discuss the quantization of the radiation field. We present important states of the field and give details of the phase-space distributions for the radiation field.

1.1 Quantization of the electromagnetic field

In this section we introduce the key features associated with the quantization of the electromagnetic field. As we will see in subsequent chapters, quantization is essential in order to understand a very wide variety of quantum optical phenomena. Let us first consider a plane electromagnetic wave of frequency ω_k that is propagating in the direction $\vec{k}$ in free space. The electric $\vec{E}$ and magnetic $\vec{B}$ fields associated with such a plane wave are given by [11]

$$\begin{aligned} \vec{E}(\vec{r},t) &= \vec{\epsilon} E_0 \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{r}-\mathrm{i}\omega_k t} + c.c., \\ \vec{B}(\vec{r},t) &= \tfrac{\vec{k}\times\vec{\epsilon}}{k} E_0 \mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{r}-\mathrm{i}\omega_k t} + c.c., \end{aligned} \tag{1.1}$$

for $k = |\vec{k}| = \omega_k/c$, where c is the velocity of light and $\vec{\epsilon}$ denotes the polarization vector for the electromagnetic field. The symbol $c.c.$ stands for the complex conjugate. We will use CGS units throughout the book. We choose especially form (1.1) as this would immediately correspond to the form used in quantum theory. The polarization vector is orthogonal to the direction of propagation and lies in a plane perpendicular to $\vec{k}$. There are two orthogonal polarizations. Let us denote these by $\vec{\epsilon}_{\vec{k}s}$, with s taking two values 1, 2. For a wave in the direction z we can write $\vec{\epsilon}$ in terms of the unit vectors along the x and y axes as

$$\vec{\epsilon}_{\vec{k}s} = \epsilon_{xs}\hat{x} + \epsilon_{ys}\hat{y}, \quad |\epsilon_{xs}|^2 + |\epsilon_{ys}|^2 = 1. \tag{1.2}$$

For real ϵ_{xs} and ϵ_{ys} we have linearly polarized light; for

$$\vec{\epsilon}_{\vec{k}s} = \frac{1}{\sqrt{2}}(\hat{x} \pm \mathrm{i}\hat{y}), \tag{1.3}$$

we have circularly polarized light. Note that the vectors $\vec{E}$, $\vec{B}$, and $\vec{k}$ form a right-handed orthogonal coordinate system. The energy of the electromagnetic field, contained in volume V, is given by

$$U \equiv \frac{1}{8\pi}\int_V [E^2(\vec{r},t) + B^2(\vec{r},t)]\,\mathrm{d}^3 r, \tag{1.4}$$

which for a plane wave reduces to

$$U \equiv \frac{1}{2\pi}|E_0|^2 V. \tag{1.5}$$

The Poynting vector $\vec{S}$, defined by

$$\vec{S} \equiv \frac{c}{4\pi}\vec{E}\times\vec{B}, \tag{1.6}$$

reduces for the plane wave to

$$\vec{S} \equiv \frac{c}{2\pi}|E_0|^2\left(\frac{\vec{k}}{k}\right). \tag{1.7}$$

In deriving (1.5) and (1.7) we have dropped the fast-oscillating terms at frequency $2\omega_k$.

In many problems it is more convenient to work with potentials, such as the vector potential $\vec{A}(\vec{r},t)$. Since the current book is devoted to problems in the nonrelativistic domain, we adopt the Coulomb gauge or transverse gauge in which the scalar potential is set to zero and $\vec{A}$ satisfies

$$\mathrm{div}\vec{A}(\vec{r},t) = 0. \tag{1.8}$$

The electric and magnetic fields are related to $\vec{A}$ via

$$\vec{E} = -\frac{1}{c}\frac{\partial\vec{A}}{\partial t}, \quad \vec{B} = \vec{\nabla}\times\vec{A}. \tag{1.9}$$

For the case of a plane wave we can write

$$\vec{A}(\vec{r}, t) = \vec{\epsilon} A_0 e^{i\vec{k}\cdot\vec{r} - i\omega_k t} + c.c.; \quad E_0 = \frac{i\omega_k}{c} A_0. \tag{1.10}$$

In quantum theory the vector potential is more fundamental than the electric and magnetic fields.

Consider next an electromagnetic field confined to a box with volume V, we expand the field into a complete set of plane waves. The complete set can be obtained by imposing boundary conditions at the walls of the box. For convenience let us take the box to be a cube with volume L^3. Then imposing periodic boundary conditions, the allowed values of $\vec{k}$ are

$$\vec{k} \equiv 2\pi \frac{\vec{n}}{L}, \quad \vec{n}_i \equiv 0, \pm 1, \pm 2, \ldots. \tag{1.11}$$

Here each component of $\vec{n}$ is an integer with all possible values. Thus we write the vector potential in the form

$$\vec{A}(\vec{r}, t) = \sum_{\vec{k}, s} \frac{A_{\vec{k}s}}{\sqrt{V}} \vec{\epsilon}_{\vec{k}s} e^{i\vec{k}\cdot\vec{r} - i\omega_k t} + c.c.. \tag{1.12}$$

The summation in (1.12) is over all allowed values of $\vec{k}$. It should be borne in mind that for each $\vec{k}$, s takes two values. The coefficients $A_{\vec{k}s}$ are arbitrary. The specific form of $A_{\vec{k}s}$ would be determined by the electromagnetic field at hand. Using (1.12) and (1.9) and the orthogonality of the plane waves for $\vec{n} = (n_x, n_y, n_z)$ and $\vec{n}' = (n'_x, n'_y, n'_z)$,

$$\frac{1}{V} \int_V e^{i\vec{k}\cdot\vec{r} - i\vec{k}'\cdot\vec{r}} d^3 r \equiv \delta_{n_x n'_x} \delta_{n_y n'_y} \delta_{n_z n'_z}, \tag{1.13}$$

we obtain from (1.4) the expression for the energy

$$U = \frac{1}{2\pi} \sum_{\vec{k}, s} \left(\frac{\omega_k^2}{c^2} \right) |A_{\vec{k}s}|^2, \quad \omega_k = kc. \tag{1.14}$$

The energy has been expressed as a sum over modes – each mode is a plane wave with a given polarization.

We would now proceed with the quantization of the field. Clearly $\hbar\omega_k$ is the quantum of energy associated with a single mode. Let $n_{\vec{k}s}$ be the number of quanta associated with each mode. Therefore the total energy would be

$$U = \sum_{\vec{k}, s} \hbar\omega_k n_{\vec{k}s}. \tag{1.15}$$

On comparison with (1.14) we can thus identify

$$\frac{\omega_k}{2\pi\hbar c^2} |A_{\vec{k}s}|^2 \leftrightarrow n_{\vec{k}s}. \tag{1.16}$$

In quantum theory all fields $\vec{E}$, $\vec{B}$, and $\vec{A}$ become operators. The energy U becomes the Hamiltonian operator. The number becomes the number operator. Each mode of the electromagnetic field can be identified with a photon. It was demonstrated by Bose [7] that photons obey what is now called Bose statistics and thus each mode can be occupied by an arbitrary number of photons. Planck had already established that for a black body at temperature T, the average occupation number $n_{\vec{k}s}$ is

$$\langle n_{\vec{k}s}\rangle = \frac{1}{\exp(\frac{\hbar\omega_k}{K_B T}) - 1}, \tag{1.17}$$

where K_B is the Boltzmann constant. Depending on the temperature and ω_k, $\langle n_{\vec{k}s}\rangle$ can take arbitrary values. In quantum theory the number operator $n_{\vec{k}s}$ is a positive definite Hermitian operator, with eigenvalues 0, 1, 2, ... It can thus be written in terms of the non-Hermitian operators $a_{\vec{k}s}$ and $a^\dagger_{\vec{k}s}$ as

$$n_{\vec{k}s} \equiv a^\dagger_{\vec{k}s} a_{\vec{k}s}. \tag{1.18}$$

The operators $a_{\vec{k}s}$ and $a^\dagger_{\vec{k}s}$ obey the Boson commutation relations

$$[a_{\vec{k}s}, a^\dagger_{\vec{k}'s'}] = \delta_{\vec{k}\vec{k}'}\delta_{ss'}, \tag{1.19}$$

$$[a_{\vec{k}s}, a_{\vec{k}'s'}] = 0. \tag{1.20}$$

The noncommutativity of a and $a^\dagger$ adds a new dimension to the electromagnetic field. This is because all field operators are linear in a and $a^\dagger$, whereas the energy is quadratic in a and $a^\dagger$. Thus the energy can be nonzero even if there are no quanta in the field. This can be seen explicitly by using $A_{\vec{k}s} \to \sqrt{\frac{2\pi\hbar c^2}{\omega_k}} a_{\vec{k}s}$ on the basis of (1.16). All quantum fields can be expressed as [12]

$$\vec{A}(\vec{r},t) = \sum_{\vec{k},s} \sqrt{\frac{2\pi\hbar c^2}{\omega_k V}} \vec{\epsilon}_{\vec{k}s} a_{\vec{k}s} e^{i\vec{k}\cdot\vec{r} - i\omega_k t} + H.c., \tag{1.21}$$

$$\vec{E}(\vec{r},t) = i\sum_{\vec{k},s} \sqrt{\frac{2\pi\hbar\omega_k}{V}} \vec{\epsilon}_{\vec{k}s} a_{\vec{k}s} e^{i\vec{k}\cdot\vec{r} - i\omega_k t} + H.c., \tag{1.22}$$

$$\vec{B}(\vec{r},t) = i\sum_{\vec{k},s} \sqrt{\frac{2\pi\hbar\omega_k}{V}} \frac{\vec{k}\times\vec{\epsilon}_{\vec{k}s}}{k} a_{\vec{k}s} e^{i\vec{k}\cdot\vec{r} - i\omega_k t} + H.c.. \tag{1.23}$$

Here $H.c.$ stands for Hermitian conjugate because we are now dealing with quantum fields, which have to be Hermitian. The expression (1.4), after we replace energy U by the corresponding operator Hamiltonian H, becomes

$$H = \sum_{\vec{k},s} \frac{1}{2}\hbar\omega_k (a^\dagger_{\vec{k}s} a_{\vec{k}s} + a_{\vec{k}s} a^\dagger_{\vec{k}s}) \tag{1.24}$$

$$= \sum_{\vec{k},s} \hbar\omega_k \left(n_{\vec{k}s} + \frac{1}{2}\right). \tag{1.25}$$

The contribution $\sum_{\vec{k},s} \hbar\omega_k/2$ is called the zero point energy of the electromagnetic field. Furthermore, in quantum theory one writes all Hermitian field operators $\vec{F} = (\vec{A}, \vec{E}, \vec{B})$ as

$$\vec{F}(\vec{r},t) = \vec{F}^{(+)}(\vec{r},t) + \vec{F}^{(-)}(\vec{r},t), \tag{1.26}$$

where $F^{(-)}$ is the adjoint of $F^{(+)}$ and $F^{(+)}$ contains only the positive frequencies ω_k. $F^{(+)}$ is called as the positive frequency part of F and consists of only the annihilation operators $a_{\vec{k}s}$. The decomposition (1.26) is related to how the photon detectors respond to the electromagnetic field (see Section 8.1). Finally, note that in dealing with free space we would take the limit $V \to \infty$ at the end of the calculation, i.e. after we have calculated the physical observables.

1.2 State space for the electromagnetic field – Fock space and Fock states

In quantum theory all observables are represented by Hermitian operators. The expectation values of such operators in the state of the quantum system gives the measureable quantities. We thus need to specify the appropriate state space for the electromagnetic field. The quantization of a system of bosons with finite mass was done by Fock. The corresponding space is called the Fock space and the basis states are called Fock states. Since all the fields (Eqs. (1.21)–(1.23)) are written as superpositions of all the independent modes, we can construct states for each mode and from this obtain states for the multimode field.

1.2.1 State space for a single mode of the radiation field

For brevity we drop the subscript $\vec{k}s$ and denote the single-mode operators as a, $a^\dagger$, and n, with

$$[a, a^\dagger] = 1. \tag{1.27}$$

The smallest eigenvalue of the number operator is zero. Let us denote the states of the number operator as $|n\rangle$

$$a^\dagger a|n\rangle = n|n\rangle; \quad n = 0, 1, 2, \ldots, \infty. \tag{1.28}$$

The states $|n\rangle$ are called Fock states. For $n = 0$, $a^\dagger a|0\rangle = 0$, therefore we can define the state $|0\rangle$ by

$$a|0\rangle = 0. \tag{1.29}$$

The state $|0\rangle$ is called the vacuum state as it contains no quanta of the radiation field. Now using the commutation relation between a and $a^\dagger$ and applying it to the vacuum state we get $(aa^\dagger - a^\dagger a)|0\rangle = |0\rangle \Rightarrow a(a^\dagger|0\rangle) = |0\rangle$, which can be rewritten as

$$a^\dagger a(a^\dagger|0\rangle) = a^\dagger|0\rangle, \tag{1.30}$$

and hence $a^\dagger|0\rangle$ is an eigenstate of $a^\dagger a$ with eigenvalue 1, i.e.

$$|1\rangle = a^\dagger|0\rangle. \tag{1.31}$$

The state is called a single-photon state. One can continue this process and obtain the n photon state $|n\rangle$ by repeated application of the operator $a^\dagger$ on $|0\rangle$. The state $|n\rangle$ is found to be

$$|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}}|0\rangle. \tag{1.32}$$

The factor $1/\sqrt{n!}$ leads to the correct normalization of the state. The set of states $|n\rangle$ are orthogonal and complete

$$\sum_{n=0}^{\infty} |n\rangle\langle n| = 1, \quad \langle n|m\rangle = \delta_{nm}, \tag{1.33}$$

and further have the important property

$$a|n\rangle = \sqrt{n}|n-1\rangle, \quad a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle. \tag{1.34}$$

The property (1.34) is proved using (1.32) and by using the commutator (1.27). Note that the operators a and $a^\dagger$ are respectively called the annihilation and creation operators. This is because the action of a [$a^\dagger$] on the n photon state $|n\rangle$ yields the $(n-1)$ [$(n+1)$] photon state.

For a multimode field we would write the states as $|\{n_{\vec{k}s}\}\rangle$, which means that the mode $\vec{k}s$ has $n_{\vec{k}s}$ photons. These states are a product of the states for each mode

$$|\{n_{\vec{k}s}\}\rangle = \prod_{\vec{k}s} |n_{\vec{k}s}\rangle, \tag{1.35}$$

and have the properties

$$\begin{aligned} a_{\vec{k}s}|\{n_{\vec{k}s}\}\rangle &= \sqrt{n_{\vec{k}s}}|n_{\vec{k}s}-1\rangle \prod_{\vec{k}'s'\neq\vec{k}s} |n_{\vec{k}'s'}\rangle, \\ a^\dagger_{\vec{k}s}|\{n_{\vec{k}s}\}\rangle &= \sqrt{n_{\vec{k}s}+1}|n_{\vec{k}s}+1\rangle \prod_{\vec{k}'s'\neq\vec{k}s} |n_{\vec{k}'s'}\rangle. \end{aligned} \tag{1.36}$$

1.3 Quadratures of the field

For a plane-wave field (1.1), the amplitude E_0 is a complex number. Thus E_0 has a phase which can be measured by using an interferometer. One can thus obtain information on both the real and imaginary parts of E_0. These are called the in-phase and out-of-phase quadratures of the field. The well-known homodyne measurement can directly yield these quadratures. In quantum theory E_0 gets replaced by the non-Hermitian annihilation operator a. We can then define the two Hermitian quadrature operators X and Y as

$$a = \frac{X + \mathrm{i}Y}{\sqrt{2}}, \quad X = \frac{a + a^\dagger}{\sqrt{2}}, \quad Y = \frac{a - a^\dagger}{\sqrt{2}\mathrm{i}}. \tag{1.37}$$

In view of (1.27) we now get

$$[X, Y] = \mathrm{i}. \tag{1.38}$$

Note that this commutation relation for the quadratures is similar to the commutation relation for the position and momentum operators of a particle with mass. The operator X should not be confused with the position of the photon. We can, however, introduce a quadrature space by representing Y as $Y = -\mathrm{i}\frac{\partial}{\partial X}$, which follows from the commutation relation (1.38). This enables us to write the Fock states in quadrature space as

$$\Psi_n(X) = \langle X|n\rangle = (2^n n!\sqrt{\pi})^{-1/2} H_n(X)\mathrm{e}^{-X^2/2}, \tag{1.39}$$

where $H_n(X)$ is the Hermite polynomial of degree n. Furthermore, the Heisenberg uncertainty relation would give

$$\Delta X \Delta Y \geq \frac{1}{2}, \quad (\Delta X)^2 = \langle X^2\rangle - \langle X\rangle^2, \quad (\Delta Y)^2 = \langle Y^2\rangle - \langle Y\rangle^2. \tag{1.40}$$

For Fock states the relation (1.40) reduces to

$$\Delta X \Delta Y = n + \frac{1}{2}, \quad (\Delta X)^2 = (\Delta Y)^2 = n + \frac{1}{2}. \tag{1.41}$$

Clearly the quadratures carry the phase-dependent information on the field and are important in the context of the characterization and detection of the squeezed states of the field.

1.4 Coherent states

In classical theory, one can have a very well-defined electromagnetic field, i.e. a field E_0 with well-defined amplitude and phase. We have seen that in quantum theory E_0 is replaced by the annihilation operator a. Thus one would clearly identify E_0 with the expectation value of the quantum field a in a given quantum-mechanical state of the field. However, for the Fock states $|n\rangle$ of the field the mean values of a and the quadrature operators vanish

$$\langle n|a|n\rangle = \langle n|X|n\rangle = \langle n|Y|n\rangle = 0. \tag{1.42}$$

Thus Fock states could not represent fields with well-defined amplitudes and phase at a classical level. We know that a field produced by a single-mode laser is coherent, i.e. has a well-defined amplitude and phase. So the question is – what is the appropriate state of the radiation field that would represent such a coherent field? Glauber [13, 14] gave the answer to such a question and derived a new class of states that he called coherent states, which are usually denoted by the symbol $|\alpha\rangle$. If the eigenvalue equation

$$a|\alpha\rangle = \alpha|\alpha\rangle, \quad \alpha \text{ complex number}, \tag{1.43}$$

has a normalizable solution $|\alpha\rangle$, then the field and its quadratures would have nonzero values

$$\langle\alpha|a|\alpha\rangle = \alpha; \quad \langle\alpha|X|\alpha\rangle = \frac{\mathrm{Re}\{\alpha\}}{\sqrt{2}}, \quad \langle\alpha|Y|\alpha\rangle = \frac{\mathrm{Im}\{\alpha\}}{\sqrt{2}}. \tag{1.44}$$

Thus the state $|\alpha\rangle$ would correspond to a classical field with well-defined amplitude and phase and hence the name coherent states is used for such states. The intensity, which is proportional to $\langle a^\dagger a\rangle$, would be

$$\langle\alpha|a^\dagger a|\alpha\rangle = |\alpha|^2 = |\langle\alpha|a|\alpha\rangle|^2. \tag{1.45}$$

This is again like a coherent classical field as then the intensity of the field is the modulus of the square of the mean amplitude of the field.

1.4.1 Solution to the eigenvalue equation (1.43)

We can expand $|\alpha\rangle$ in terms of the Fock states as these form a complete set

$$|\alpha\rangle = \sum_{n=0}^{\infty} c_n|n\rangle, \tag{1.46}$$

which on substituting in (1.43) yields the recursion relation

$$c_{n+1} = \frac{\alpha}{\sqrt{n+1}}c_n, \tag{1.47}$$

whose solution is

$$c_n = \frac{\alpha^n}{\sqrt{n!}}c_0, \tag{1.48}$$

with c_0 fixed by the normalization condition $\langle\alpha|\alpha\rangle = 1 \Rightarrow c_0 = \exp(-\frac{1}{2}|\alpha|^2)$. Thus for all complex values of α we have the solution

$$|\alpha\rangle = \mathrm{e}^{-\frac{1}{2}|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{1.49}$$

The coherent states are superpositions of Fock states. The probability p_n of finding the system in the state $|n\rangle$ is then given by

$$p_n = |c_n|^2 = \mathrm{e}^{-|\alpha|^2}\frac{(|\alpha|^2)^n}{n!}. \tag{1.50}$$

The probability of finding n photons in a coherent state is given by the Poisson distribution (1.50) with mean $|\alpha|^2$. This distribution is shown in Figure 1.1 and has the property that its variance is equal to the mean

$$\langle n^2\rangle - \langle n\rangle^2 = \langle n\rangle; \quad \langle n\rangle = |\alpha|^2, \tag{1.51}$$

where n is the number operator $a^\dagger a$.

1.4.2 Properties of coherent states

Next we present some important properties of coherent states. The states form a complete set

$$\frac{1}{\pi}\int \mathrm{d}^2\alpha|\alpha\rangle\langle\alpha| = 1, \quad \alpha = x + \mathrm{i}y, \quad \mathrm{d}^2\alpha = \mathrm{d}x\,\mathrm{d}y, \quad -\infty \le x, y \le +\infty. \tag{1.52}$$

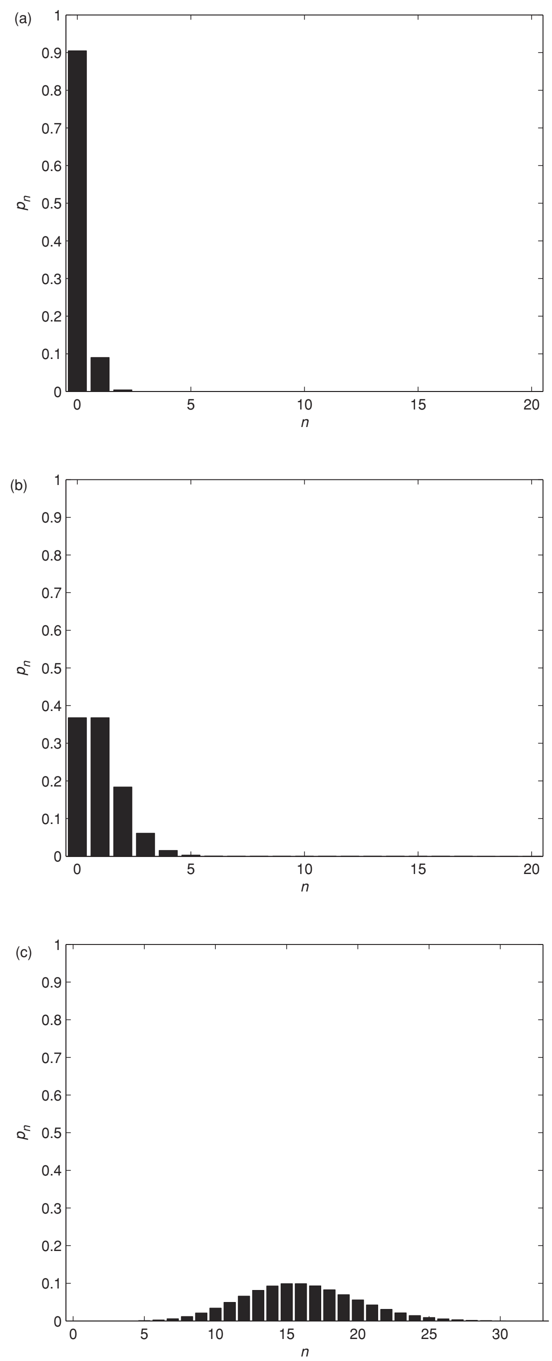

Fig. 1.1 The photon distribution p_n for a coherent state with different values of $|\alpha|^2$. (a) $|\alpha|^2 = 0.1$; (b) $|\alpha|^2 = 1$; and (c) $|\alpha|^2 = 16$.

The proof proceeds as follows – we substitute (1.49) into (1.52), hence the LHS becomes

$$\sum_{n,m} \frac{1}{\sqrt{n!m!}} \int d^2\alpha \, \alpha^n \alpha^{*m} |n\rangle\langle m| e^{-|\alpha|^2}.$$

The integral is easily evaluated in polar coordinates, $\alpha = r e^{i\theta}$,

$$\sum_{n,m} \frac{1}{\sqrt{n!m!}} \int_0^\infty r dr \int_0^{2\pi} d\theta \, (r)^{n+m} e^{i\theta(n-m)} |n\rangle\langle m| e^{-r^2}.$$

The θ integral gives δ_{nm}, and the r integral is evaluated in terms of the gamma function, yielding

$$\sum_n |n\rangle\langle n| = 1. \qquad \square$$

The coherent states are nonorthogonal

$$\langle\alpha|\beta\rangle \equiv \exp\left(\alpha^*\beta - \frac{1}{2}|\alpha|^2 - \frac{1}{2}|\beta|^2\right), \tag{1.53}$$

which follows in a straightforward manner by substituting (1.49) into (1.53). In view of (1.52) into (1.53), the function (1.53) is an example of a reproducing kernel $K(\alpha, \beta)$

$$\int d^2\beta \, K(\alpha, \beta) K(\beta, \gamma) = K(\alpha, \gamma), \quad K(\alpha, \beta) = \langle\alpha|\beta\rangle/\pi. \tag{1.54}$$

The nonorthogonality of coherent states leads to the unusual property that a given coherent state can be written in terms of the other coherent states

$$|\alpha\rangle = \int K(\alpha, \beta)|\beta\rangle \, d^2\beta. \tag{1.55}$$

The coherent states can be generated by displacing the vacuum

$$|\alpha\rangle = D(\alpha)|0\rangle, \tag{1.56}$$

where $D(\alpha)$, called the displacement operator, is defined by

$$D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a). \tag{1.57}$$

The relation (1.56) is important as it shows how the coherent states can be generated. To prove (1.56) we use the Baker–Campbell–Hausdorff identity [15] to write (1.56) as

$$\begin{aligned} |\alpha\rangle &= e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^\dagger} e^{-\alpha^* a} |0\rangle \\ &= e^{-\frac{1}{2}|\alpha|^2} e^{a^\dagger \alpha} |0\rangle, \quad \text{as} \quad a^n|0\rangle = 0 \\ &= e^{-\frac{1}{2}|\alpha|^2} \sum_{n=0}^\infty \frac{\alpha^n}{n!} (a^\dagger)^n |0\rangle, \end{aligned} \tag{1.58}$$

which leads to (1.49) by using the definition (1.32). The displacement operators are quite important in many calculations with coherent states. We list in Table 1.1 many of their mathematical properties.