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Light beams carrying orbital angular momentum

J. B. GÖTTE AND S. M. BARNETT

For optical fields the notion of a total angular momentum has long been known. The concept of a light beam carrying orbital angular momentum, however, was unfamiliar until it was discovered that Laguerre–Gaussian beams, within the paraxial approximation, carry a well-defined orbital angular momentum [1, 2]. This discovery started the modern interest in orbital angular momentum of light. In this chapter we discuss the theoretical framework of orbital angular momentum of light in terms of fields and light beams and how to generate these. The material in this chapter is based in parts on the PhD thesis of Götte [3].

1.1 Introduction

A quantitative treatment of the mechanical effects of light became possible only after light had been integrated into Maxwell's dynamical theory of electromagnetic waves. With this theory Poynting [4] derived a continuity equation for the energy in the electromagnetic field. After Heaviside [5, 6] introduced the vectorial notation for the Maxwell equations this continuity equation could be written in its modern form using the Poynting vector. Interestingly, the linear momentum density in the electromagnetic field is also given by the Poynting vector apart from constant factors depending on the chosen system of units. Poynting [7] also derived an expression for the angular momentum of circularly polarised light by means of a mechanical analogue in the form of a rotating shaft. Later, Poynting's expression was verified by measuring the torque on a quarter wave-plate due to circularly polarised light [8]. By this time quantum mechanics had been firmly established and Beth [8] showed that the same quantitative result could be obtained whether the torque was calculated classically, or from the assumption that each quantum of circularly polarised light carries an angular momentum of $\pm\hbar$.

While spin angular momentum in light beams was thus established in terms of circularly polarised light, the orbital angular momentum of light beams remained a less familiar concept. This does not mean, however, that the orbital angular momentum of light fields

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in general has not been studied [9]. The total angular momentum density is given by the cross product of the radial vector and the linear momentum density, which is a form typical for orbital angular momentum [10]. This intriguing fact shows the difficulties in splitting the total angular momentum into spin and orbital parts for a general electromagnetic field. It was discovered at Leiden University that Laguerre–Gaussian light beams, familiar from paraxial optics or laser theory and realisable in a laboratory, have a well-defined separation of the total angular momentum into spin and orbital parts [1, 2]. Both can be transferred separately to a trapped particle causing a rotation around the centre of the particle, owing to spin angular momentum, or around the centre of the beam owing to orbital angular momentum [11]. This has led to a wide use of the angular momentum of light in the field of optical micromanipulation and trapping [12, 13].

The separation into spin and orbital angular momentum is also crucial for the quantum mechanical properties of light. The two orthogonal polarisations, for example left and right circularly polarised light, and thus the spin angular momentum have long been used as an optical implementation of a qubit in quantum information [14]. The possibility to produce pairs of photons entangled in both spin and orbital angular momentum [15, 16] has created a strong interest in optical angular momentum as a resource in quantum information. In contrast to spin angular momentum, orbital angular momentum is not restricted to two orthogonal states, which is why it is used as a high-dimensional quantum system (see Chapters 12 and 16).

In this chapter we review the concept of orbital angular momentum of light beams. Starting from the Maxwell equations we look at the continuity equation for linear and angular momentum of the electromagnetic field. We derive the exact Helmholtz equation and then the paraxial approximation to it in order to study light beams with orbital angular momentum in the paraxial and non-paraxial regime. This allows us to establish that light beams with a particular azimuthal phase structure have orbital angular momentum. At the end of this chapter we will describe some methods to generate these beams in a laboratory.

1.2 Mechanical properties of optical fields

The set of Maxwell equations describes the electromagnetic field including its mechanical properties. The conservation of mechanical properties such as energy, linear and angular momentum are expressed by continuity equations. These equations describe the change of quantities in a continuum and are therefore stated in terms of densities. The rate of change for a density ρ of a locally conserved quantity is given by the divergence of the respective flux density $\mathbf{j}$ and the source density q:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = q. \tag{1.1}$$

Here, ρ could stand for the density of electromagnetic charge or energy as well as the density of linear or angular momentum, while $\mathbf{j}$ and q are the corresponding flux and source densities respectively. In case a quantity is locally conserved the source density vanishes identically.

The relevant continuity equation for the study of optical angular momentum governs the local conversation of the total angular momentum. However, for the electromagnetic field the continuity equations for the mechanical quantities are intricately linked; the flux density of the energy is also the density for the linear momentum and the angular momentum densities are directly related to the linear momentum densities. In the following we therefore review the continuity equations for energy, and the linear and angular momentum of the electromagnetic field.

1.2.1 Energy of the electromagnetic field

Electromagnetic energy can be converted into mechanical or thermal energy. This conversion of energy will be mediated through work exerted on a charge q through the Lorentz force [17, 18]

$$\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right), \tag{1.2}$$

which is stated here for a point charge q with velocity $\mathbf{v}$. In this chapter we will use vectorial notation along with notation in components. The Cartesian component of a vector $\mathbf{a}$ will be denoted by a_i, where $i = 1, 2, 3$ represents the x, y or z component respectively. The Cartesian components of the Lorentz force may thus be we written as

$$F_i = q\left(E_i + \epsilon_{ijk} v_j B_k\right), \tag{1.3}$$

where we have used the Levi–Civita symbol ϵ_{ijk} to write the cross product $\mathbf{v} \times \mathbf{B}$ in components. The Levi–Civita symbol is equal to 1 if the combination of the indices ijk is an even permutation of 123 and it is equal to -1 for an odd permutation. Otherwise it is zero. We also use the convention of implicitly summing over doubly occurring indices.

The work exerted on a point charge between two arbitrary times t_0 and t_1 is thus given by

$$W(t_0, t_1) = \int_{t_0}^{t_1} \mathrm{d}t\ v_i F_i = q \int_{t_0}^{t_1} \mathrm{d}t\ v_i E_i. \tag{1.4}$$

The magnetic field does not contribute to the work as the cross product $\mathbf{v} \times \mathbf{B}$ is perpendicular to $\mathbf{v}$. For a continuous distribution of charges ρ in a velocity field $\mathbf{v}$ the energy source density of the electromagnetic fields is expressed as [10]

$$q^E = -j_i E_i = -\mathbf{j} \cdot \mathbf{E}, \tag{1.5}$$

where the charge flux density $\mathbf{j}$ is given by $\rho\mathbf{v}$. The work exerted reduces the energy of the electric field, which explains the minus sign in the expression above. The expression in Eq. (1.5) has the dimension of power which is lost through work done on the charges. The loss in power is balanced by the rate of change of the energy of the electromagnetic field and the energy flux. In the absence of any charged particles, however, the continuity equation has no source density term and the rate of change of the energy density is balanced by the divergence of the energy density flux only.

To find the expressions for the energy density and the energy flux density we can use the Maxwell equations for external charge and flux densities [18, 19]

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \tag{1.6a}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{1.6b}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{1.6c}$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{j} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right). \tag{1.6d}$$

On substituting Eq. (1.6d) into Eq. (1.5) and using Eq. (1.6b) to substitute $\nabla \times \mathbf{E}$ we find that

$$-\mathbf{j} \cdot \mathbf{E} = \frac{\partial}{\partial t} \left(\frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2 \right) + \nabla \cdot \left(\frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \right). \tag{1.7}$$

This has the required form of a continuity equation for the energy of the electromagnetic field if we identify

$$\rho^E \equiv w = \frac{\epsilon_0}{2} E^2 + \frac{1}{2\mu_0} B^2 \tag{1.8}$$

with the energy density of the electromagnetic field and

$$\mathbf{j}^E \equiv \mathbf{S} = \frac{1}{\mu_0} \mathbf{E} \times \mathbf{B} \tag{1.9}$$

with the energy flux density. The quantity $\mathbf{S}$ is often called the Poynting vector as it was Poynting [4] who discovered the law for the transfer of energy in the electromagnetic field. The continuity equation is therefore also known as Poynting's theorem. A continuity equation does not determine the density and flux density completely; the continuity equation Eq. (1.1) is also fulfilled for a density ρ^E augmented by the divergence $\nabla \cdot \mathbf{V}$ of an arbitrary vector field $\mathbf{V}$ if its time derivative $\partial \mathbf{V}/\partial t$ is subtracted from the flux density $\mathbf{j}^E$. If $\mathbf{V}$ falls off sufficiently quickly, both expressions for the energy density give rise to the same total energy. In addition it is always possible to add the curl $\nabla \times \mathbf{W}$ of an arbitrary vector field $\mathbf{W}$ to the energy flux density $\mathbf{j}^E$ as only the divergence of the flux density enters the continuity equation [20]. It is, however, possible to determine the energy density uniquely from relativistic considerations [18, 21] and these suggest that we should set $\mathbf{V}$ and $\mathbf{W}$ equal to zero. Finally, it should be noted that the Poynting vector in Eq. (1.9) is not a generally valid expression for the energy flux density, but rather only in the special case of external sources in vacuum, where the electric displacement $\mathbf{D}$ is related to the electric field $\mathbf{E}$ by $\mathbf{D} = \epsilon_0 \mathbf{E}$ and ϵ_0 is the free space electric permittivity. Also, the magnetic induction $\mathbf{B}$ has to be related to the magnetic field $\mathbf{H}$ by $\mathbf{B} = \mu_0 \mathbf{H}$ and μ_0 is the free space magnetic permittivity. For the propagation of light beams we will treat the electromagnetic wave as a closed system without any sources. In this case the expression for the energy flux density is rigorous [22].

1.2.2 Linear momentum of the electromagnetic field

The linear momentum is a vector and we will use the notation in components to express the continuity equation for a vector quantity:

$$\frac{\partial \rho_k^P}{\partial t} + \frac{\partial j_{ik}^P}{\partial x_i} = q_k^P, \tag{1.10}$$

where the superscript P denotes the locally preserved quantity, in this case the momentum. For the linear momentum the rate of change of the momentum density and the divergence of the momentum flux density are balanced by the momentum source density. It is not surprising that the source density is given by minus the Lorentz force density:

$$q_k^P \equiv f_k = -\rho E_k - \epsilon_{klm} j_l B_m. \tag{1.11}$$

On using the Maxwell equations (1.6) to manipulate the expression for the source density the continuity equation for the linear momentum can be derived if we identify

$$\rho_k^P \equiv \frac{1}{c^2}\mu_0 S_k = \epsilon_0 \epsilon_{klm} E_l B_m \tag{1.12}$$

with the linear momentum density. This is identical to the energy flux density apart from the factor μ_0/c^2. The momentum flux density is given by

$$j_{ik}^P \equiv -\mathcal{T}_{ik} = \epsilon_0 \left(\frac{1}{2} E^2 \delta_{ik} - E_i E_k \right) + \frac{1}{\mu_0} \left(\frac{1}{2} B^2 \delta_{ik} - B_i B_k \right), \tag{1.13}$$

where $\mathcal{T}$ is the symmetric energy-stress tensor [18].

For a symmetric momentum flux density the angular momentum continuity equation may be fulfilled with a density, flux density and source density which are given by a vector product of the radial position and the respective linear density. This is similar to a rigid body where the angular momentum is solely due to orbital angular momentum which is given by the cross product $\mathbf{L} = \mathbf{r} \times \mathbf{P}$ and where the torque $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{F}$ is completely determined by the force $\mathbf{F}$. In a continuum theory this indicates that there is no intrinsic angular momentum and no intrinsic torque, and that for sufficiently small volumes all angular quantities are already given by the corresponding linear quantity. For classical fluids this a well justified assumption, but circularly polarised light carries spin angular momentum [10]. This indicates the difficulties in assigning the terms ‘spin’ and ‘orbital’ to specific parts of the total angular momentum. For light beams this difficulty may be resolved, at least in part, by using the angular momentum flux rather than the angular momentum density itself [23].

1.2.3 Angular momentum of the electromagnetic field

The density, flux density and source density for the angular momentum are directly related to the respective linear quantities [10]:

$$\rho_k^J = \epsilon_{klm} x_l \rho_m^P, \tag{1.14a}$$

$$j_{ik}^J = \epsilon_{ijl} x_j j_{lk}^P, \tag{1.14b}$$

$$q_k^J = \epsilon_{klm} x_l q_m^P. \tag{1.14c}$$

The given angular quantities fulfil the angular momentum continuity equation owing to the symmetry of the linear momentum flux density j_{lk}^P. This is because

$$\frac{\partial}{\partial x_k} j_{ik}^J = \epsilon_{ijl} x_j \frac{\partial}{\partial x_k} j_{lk}^P, \tag{1.15}$$

as $\epsilon_{ijl} j_{lj}^P = 0$ by virtue of the symmetry in l and j of j_{lj}^P. The whole continuity equation for the angular momentum can thus be written as

$$\epsilon_{ijk} x_j \left(\frac{\partial}{\partial t} \rho_k^P + \frac{\partial}{\partial x_l} j_{kl}^P - q_k^P \right) = 0, \tag{1.16}$$

which is always fulfilled because of the validity of the continuity equation for linear momentum.

For the purpose of this chapter we will be concerned with the Maxwell equations in free space, so that all the source densities are equal to zero. The density of the optical angular momentum is

$$\boldsymbol{\rho}^J = \epsilon_0 \, \mathbf{r} \times (\mathbf{E} \times \mathbf{B}) \quad \text{or} \quad \rho_i^J = \epsilon_0 \left(E_i x_j B_j - B_i x_j E_j \right), \tag{1.17}$$

and we introduce $\mathcal{M}$ for the angular momentum flux density which has the components

$$\mathcal{M}_{ik} \equiv j_{ik}^J = \epsilon_{ilk} x_l w - \epsilon_{ilm} x_l \left(\epsilon_0 E_m E_k + \frac{1}{\mu_0} B_m B_k \right), \tag{1.18}$$

where w is the energy density from Eq. (1.8). For a light beam with an optical axis along the z direction the angular momentum flux through a plane of constant z is given by the integral

$$M_{zz} \equiv M_{33} = \iint \mathrm{d}x \, \mathrm{d}y \, \mathcal{M}_{33}. \tag{1.19}$$

With the help of these expressions, we are in the position to calculate the angular momentum of light beams and its separation into spin and orbital parts.

1.2.4 Spin and orbital angular momentum

It is natural, and also essential for the purpose of this book, to try to separate the angular momentum into spin and orbital parts. The former we shall associate with optical polarization and the latter with angular momentum derived from the phase structure of our light beams. In terms of the field, we require decomposition of the total angular momentum of the form

$$\mathbf{J} = \int \mathrm{d}V \epsilon_0\, \mathbf{r} \times (\mathbf{E} \times \mathbf{B}) = \mathbf{L} + \mathbf{S}, \tag{1.20}$$

where $\mathbf{L}$ is the orbital part and $\mathbf{S}$ is the spin part. Perhaps the most obvious way we might make this separation is to write $\mathbf{B}$ in terms of the vector potential $\mathbf{A}$ and then after a little manipulation find [24]

$$\begin{aligned}\mathbf{J} &= \epsilon_0 \int \mathrm{d}V\, [E_i(\mathbf{r} \times \nabla)A_i + \mathbf{E} \times \mathbf{A} - \nabla_i(E_i \mathbf{r} \times \mathbf{A})] \\ &= \epsilon_0 \int \mathrm{d}V\, [E_i(\mathbf{r} \times \nabla)A_i + \mathbf{E} \times \mathbf{A}] - \epsilon_0 \int (\mathbf{r} \times \mathbf{A})\mathbf{E} \cdot \mathrm{d}\mathbf{S},\end{aligned} \tag{1.21}$$

where we have used Gauss' theorem. If the fields fall off sufficiently quickly then the surface integral will be zero and we can identify the orbital and spin parts as

$$\mathbf{L} = \epsilon_0 \int \mathrm{d}V\, E_i(\mathbf{r} \times \nabla)A_i, \tag{1.22}$$

$$\mathbf{S} = \epsilon_0 \int \mathrm{d}V\, \mathbf{E} \times \mathbf{A}. \tag{1.23}$$

The analogy with the operator for angular momentum in quantum theory suggests the association of the first part with the orbital angular momentum. The second part, however, clearly depends on the vectorial nature of the field and hence its polarization, which makes it natural to associate it with the spin.

The appearance of the vector potential in $\mathbf{L}$ and $\mathbf{S}$ has caused much confusion and it has been suggested, for this reason, that the separation might not be physically meaningful [24–28]. It is now known, however, that the separation is meaningful although neither part, $\mathbf{L}$ nor $\mathbf{S}$, is itself an angular momentum, as defined by the commutation relations [29, 30]. The reason for this can be traced to the impossibility of separately rotating either the direction of the fields or their spatial distribution without violating transversality [31].

It is perhaps interesting to mention one further subtlety associated with the orbital and spin parts of the angular momentum. In the absence of charges and currents, Maxwell's equations are invariant under the Heaviside–Lamor or duplex transformation [18, 32, 33].

$$\mathbf{E} \longrightarrow \cos\theta\, \mathbf{E} + \sin\theta\, c\mathbf{B}, \tag{1.24}$$

$$\mathbf{B} \longrightarrow \cos\theta\, \mathbf{B} - \sin\theta\, \frac{1}{c}\mathbf{E}. \tag{1.25}$$

All physically significant quantities for the free field, such as ρ^E, $\boldsymbol{\rho}^P$ and $\boldsymbol{\rho}^J$, are necessarily also invariant under this transformation. We can write $\mathbf{L}$ and $\mathbf{S}$ in a form which manifestly obeys this symmetry by introducing a second vector potential, $\mathbf{C}$, for which [34]

$$\mathbf{E} = -c\,\nabla \times \mathbf{C}, \tag{1.26}$$

$$\mathbf{B} = -\frac{1}{c}\frac{\partial}{\partial t}\mathbf{C}. \tag{1.27}$$

The Heaviside–Larmor symmetry then requires that quantities of physical interest should be invariant under

$$\mathbf{A} \longrightarrow \cos\theta\,\mathbf{A} + \sin\theta\,\mathbf{C}, \tag{1.28}$$

$$\mathbf{C} \longrightarrow \cos\theta\,\mathbf{C} - \sin\theta\,\mathbf{A}. \tag{1.29}$$

This allows us to write $\mathbf{L}$ and $\mathbf{S}$ in the form [31]

$$\mathbf{L} = \frac{\epsilon_0}{2}\int \mathrm{d}V \left(E_i(\mathbf{r}\times\nabla)A_i + \frac{1}{c}B_i(\mathbf{r}\times\nabla)C_i \right), \tag{1.30}$$

$$\mathbf{S} = \frac{\epsilon_0}{2}\int \mathrm{d}V \left(\mathbf{E}\times\mathbf{A} + \frac{1}{c}\mathbf{B}\times\mathbf{C} \right). \tag{1.31}$$

It is straightforward to show that these have the same value as Eqs. (1.22) and (1.23). They do suggest, however, different values for the *densities* of the orbital and spin parts of the angular momentum.

1.3 Wave equations

In this section we derive the Helmholtz equation and its paraxial approximation for electromagnetic waves in free space. For a consistent application of the paraxial approximation the wave equation has to be derived for the vector potential rather than the electric field or the magnetic induction.

1.3.1 Helmholtz equation

We are mostly concerned with the propagation of waves carrying orbital angular momentum in free space and so we do not consider the presence of charge densities ρ or flux densities $\mathbf{j}$ in the Maxwell equations. In this case the Maxwell equations are a set of homogenous differential equations for the electric field $\mathbf{E}$ and the magnetic induction $\mathbf{B}$:

$$\nabla\cdot\mathbf{E} = 0, \tag{1.32a}$$

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \tag{1.32b}$$

$$\nabla\cdot\mathbf{B} = 0, \tag{1.32c}$$

$$\nabla\times\mathbf{B} = \epsilon_0\mu_0\frac{\partial\mathbf{E}}{\partial t}. \tag{1.32d}$$

On taking the curl of either Eq. (1.32b) or Eq. (1.32d) and substituting the other equation it is possible to derive two wave equations for $\mathbf{E}$ and $\mathbf{B}$. For the electric field the double curl $\nabla \times (\nabla \times \mathbf{E})$ may be replaced by $\nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E}$ and analogously for the magnetic induction $\mathbf{B}$. In both cases the first term is equal to zero because of the Maxwell equations (1.32a) and (1.32c). We are then left with the wave equations

$$\nabla^2\mathbf{E} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{E} = 0, \tag{1.33a}$$

$$\nabla^2\mathbf{B} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{B} = 0. \tag{1.33b}$$

In the following we introduce complex electric and magnetic fields and we limit our attention to monochromatic beams with angular frequency ω. This allows us to separate the electric and magnetic field into a part with the spatial dependence and a time dependent factor $\exp(-\mathrm{i}\omega t)$:

$$\mathbf{E} = \mathrm{Re}[\boldsymbol{\mathcal{E}} \exp(-\mathrm{i}\omega t)] \quad \text{and} \quad \mathbf{B} = \mathrm{Re}[\boldsymbol{\mathcal{B}} \exp(-\mathrm{i}\omega t)]. \tag{1.34}$$

By substituting this separated ansatz into the wave equations (1.33) we obtain the Helmholtz equations for $\boldsymbol{\mathcal{E}}$ and $\boldsymbol{\mathcal{B}}$:

$$\nabla^2\boldsymbol{\mathcal{E}} + k^2\boldsymbol{\mathcal{E}} = 0, \tag{1.35a}$$

$$\nabla^2\boldsymbol{\mathcal{B}} + k^2\boldsymbol{\mathcal{B}} = 0, \tag{1.35b}$$

where $k = \omega/c$ is the overall wavenumber.

As detailed by Lax *et al.* [35], the paraxial approximation to the Helmholtz equation (1.35a), as applied to a single polarization, is inconsistent with the Maxwell equation (1.32a) and needs to be treated with care. To avoid inconsistencies it is possible to use a Helmholtz equation for the vector potential $\mathbf{A}$ [36] which is not required to have a vanishing divergence in the Lorentz gauge. The vector potential is related to the magnetic induction by

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{1.36}$$

On substituting the definition of the vector potential into Faraday's law in Eq. (1.32b) we find that the curl of the combined vector field $\mathbf{E} + (\partial\mathbf{A}/\partial t)$ is equal to zero:

$$\nabla \times \left(\mathbf{E} + \frac{\partial}{\partial t}\mathbf{A}\right) = 0. \tag{1.37}$$

This is an indication for the existence of a scalar potential Φ defined as

$$\nabla\Phi = -\left(\mathbf{E} + \frac{\partial}{\partial t}\mathbf{A}\right). \tag{1.38}$$

The vector potential and the scalar potential are not completely determined by the electric field and the magnetic induction. The derivative of a scalar function ζ may be added to the vector potential if the time-derivative of ζ is subtracted from the scalar potential [18]:

$$\mathbf{A} \longrightarrow \mathbf{A}' = \mathbf{A} + \nabla\zeta, \quad \Phi \longrightarrow \Phi' = \Phi - \frac{\partial}{\partial t}\zeta, \tag{1.39}$$

where ζ is an arbitrary function of $\mathbf{r}$ and t. The most convenient gauge for our purposes is the Lorentz gauge, as the Coulomb gauge requires that the divergence of the vector potential is equal to zero, which leads to unnecessary complications when making the paraxial approximation [37]. The Lorentz gauge is given by the condition

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\Phi = 0. \tag{1.40}$$

This also requires that the scalar function ζ obeys a wave equation:

$$\nabla^2\zeta - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\zeta = 0. \tag{1.41}$$

With the help of the Eqs. (1.36) and (1.38) it is possible to substitute the electric field and the magnetic induction in the Maxwell equations with the scalar and vector potential. The substitution satisfies the Maxwell equations (1.32b) and (1.32c) identically. The dynamical behaviour of $\mathbf{A}$ and Φ is determined by the two remaining Maxwell equations (1.32a) and (1.32d). Together with the Lorentz condition in Eq. (1.40) the two remaining Maxwell equations yield two wave equations for the scalar and the vector potential:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\Phi = 0, \tag{1.42a}$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\mathbf{A} = 0. \tag{1.42b}$$

From these, the Helmholtz equation for the vector potential is obtained with the ansatz $\mathbf{A} = \mathrm{Re}[\boldsymbol{\mathcal{A}}\exp(-\mathrm{i}\omega t)]$ for a monochromatic wave. With $k = \omega/c$ this yields the following equation:

$$\nabla^2\boldsymbol{\mathcal{A}} + k^2\boldsymbol{\mathcal{A}} = 0. \tag{1.43}$$

Once a solution for the vector potential has been found, the electric field can be calculated from the vector potential via

$$\boldsymbol{\mathcal{E}} = \mathrm{i}\omega\left(\boldsymbol{\mathcal{A}} + \frac{\nabla(\nabla\cdot\boldsymbol{\mathcal{A}})}{k^2}\right). \tag{1.44}$$

For this relation we have assumed the monochromatic form for the scalar and vector potential. The given equation follows then from the definition of the scalar potential in Eq. (1.38) and the Lorentz condition in Eq. (1.40).