Cambridge University Press
978-1-107-00620-1 - Advanced Topics in Applied Mathematics: For Engineering and the Physical Sciences
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Excerpt
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GREEN'S FUNCTIONS

Before we introduce the **Green's functions**, it is necessary to familiarize ourselves with the idea of **generalized functions** or **distributions**. These are called generalized functions as they do not conform to the definition of functions. They are often unbounded and discontinuous. They are characterized by their integral properties as linear functionals.

1.1 HEAVISIDE STEP FUNCTION

Although this is a simple discontinuous function (not a generalized function), the **Heaviside step function** is a good starting point to introduce generalized functions. It is defined as

$$h(x) = \begin{cases} 0, & x < 0, \\ 1/2, & x = 0, \\ 1, & x > 0. \end{cases} \tag{1.1}$$

The value of the function at $x = 0$ is seldom needed as we always approach the point $x = 0$ either from the right or from the left (see Fig. 1.1). When we consider representation of this function using, say, Fourier series, the series converges to the mean of the right and left limits if there is a discontinuity. Thus, $h(0) = 1/2$ will be the converged result for such a series.

Using the Heaviside function, we can express the **signum** function, which has a value of 1 when the argument is positive, and a value of -1

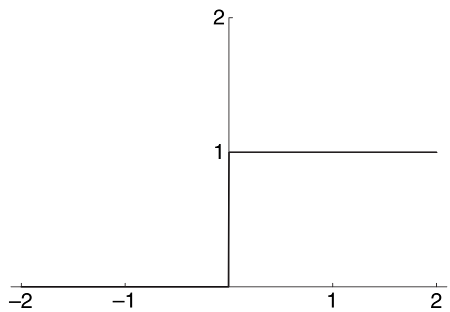

Figure 1.1. Heaviside step function.

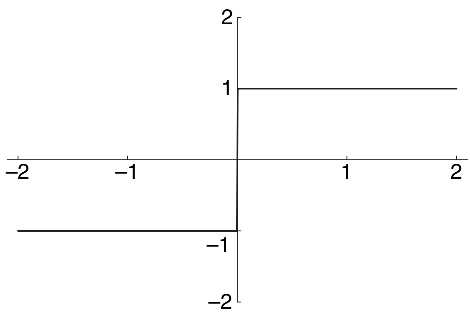

Figure 1.2. Signum function sgn(x).

when the argument is negative (see Fig. 1.2), as

$$\operatorname{sgn}(x) = 2h(x) - 1. \tag{1.2}$$

We may convert an even function of x to an odd function simply by multiplying by sgn(x).

The function shown in Fig. 1.3 can be written as

$$f(x) = h(a - |x|). \tag{1.3}$$

This is known as the **Haar function**, which plays an important role in image processing as a basis for **wavelet** expansions. In wavelet analysis,

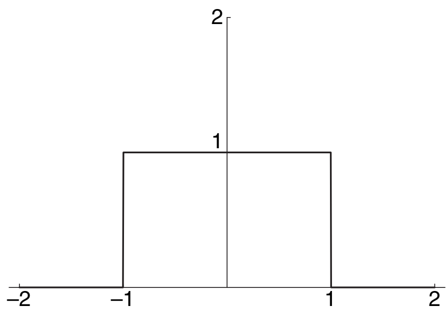

Figure 1.3. Haar function ($a = 1$).

families of Haar functions with support a, $a/2$, $a/4, \ldots, a/2^n$ are used as a basis to represent functions.

1.2 DIRAC DELTA FUNCTION

The Dirac delta function has its origin in the idea of *concentrated* charges in electromagnetics and quantum mechanics. In mechanics, the Dirac delta $\delta(x)$ is useful in representing concentrated forces. We can view this generalized function as the derivative of the Heaviside function, which is zero everywhere except at the origin. At the origin it is infinity. As a consequence, its integral from $-\epsilon$ to $+\epsilon$ is unity. As is the case for all generalized functions, we consider the delta function as the limit of various sequences of functions. For example, consider the sequence of functions shown in Fig. 1.4, which depends on the parameter ϵ,

$$f(x;\epsilon) = \begin{cases} 0, & |x| > \epsilon, \\ \frac{1}{2\epsilon}, & |x| < \epsilon. \end{cases} \tag{1.4}$$

In the limit $\epsilon \to 0, f(x;\epsilon) \to \delta(x)$.

Note that

$$\int_{-\infty}^{\infty} f(x;\epsilon)\,dx = \int_{-\epsilon}^{\epsilon} f(x;\epsilon)\,dx = 1 = \int_{-\infty}^{\infty} \delta(x)\,dx = \int_{-\epsilon}^{\epsilon} \delta(x)\,dx. \tag{1.5}$$

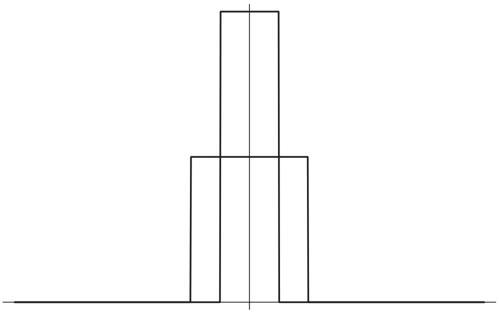

Figure 1.4. A delta sequence using Haar functions.

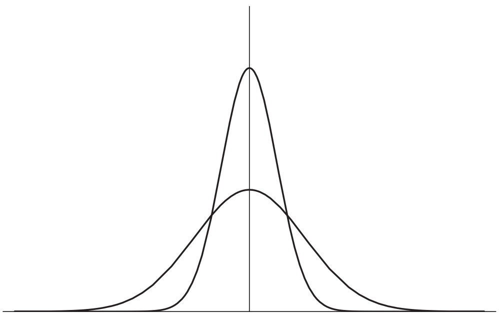

Figure 1.5. Another delta sequence using probability functions.

Another sequence of continuous functions which forms a delta sequence is given (see Fig. 1.5) by the Gauss functions or probability functions:

$$f(x;n) = \frac{n}{\sqrt{\pi}} e^{-n^2x^2}. \tag{1.6}$$

We can see that the area under this curve remains unity for all values of n. Let

$$I = \int_{-\infty}^{\infty} ne^{-n^2x^2}\, dx = \int_{-\infty}^{\infty} e^{-x^2}\, dx, \tag{1.7}$$

where we substituted $nx \to x$. Using polar coordinates,

$$I^2 = \int_{-\infty}^{\infty} e^{-x^2}\,dx \int_{-\infty}^{\infty} e^{-y^2}\,dy$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dxdy$$
$$= \int_0^{\infty}\int_0^{2\pi} e^{-r^2}\,r\,drd\theta$$
$$= 2\pi \int_0^{\infty} e^{-r^2}\,r\,dr = -\pi e^{-r^2}\Big|_0^{\infty} = \pi. \tag{1.8}$$

We frequently encounter the integral I, which has the value

$$I = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}. \tag{1.9}$$

As $n \to \infty, f(x;n) \to \delta(x)$.

By shifting the origin from $x = 0$ to $x = \xi$, we can move the spike of the delta function to the point ξ. This new function has the properties,

$$\delta(x-\xi) = 0, \quad x \neq \xi, \tag{1.10}$$

$$\int_{\xi-\epsilon}^{\xi+\epsilon} \delta(x-\xi)\,dx = 1. \tag{1.11}$$

An important property of the delta function is localization under integration. As usual, properties of the generalized functions are proved using the corresponding sequences. For any smooth function $\phi(x)$, which is nonzero only in a finite interval (a,b), using the sequence (1.4), we have

$$\int_{-\infty}^{\infty} \phi(x)\delta(x-\xi)\,dx$$
$$= \lim_{\epsilon\to 0}\int_{\xi-\epsilon}^{\xi+\epsilon} \phi(x)\,\frac{dx}{2\epsilon}$$
$$= \lim_{\epsilon\to 0}\int_{\xi-\epsilon}^{\xi+\epsilon} \left[\phi(\xi)+\phi'(\xi)(x-\xi)+\frac{1}{2}\phi''(\xi)(x-\xi)^2+\cdots\right]\frac{dx}{2\epsilon}$$

$$
\begin{aligned}
&= \lim_{\epsilon \to 0} \left[\phi(\xi) + \frac{1}{2}\phi'(\xi)\epsilon + \frac{1}{6}\phi''(\xi)\epsilon^2 + \cdots \right] \\
&= \phi(\xi). \qquad (1.12)
\end{aligned}
$$

Integrals involving a *scaled* delta function can be evaluated as shown:

$$
\begin{aligned}
\int_{-\infty}^{\infty} \phi(x)\delta(\frac{x-\xi}{a})\,dx &= \int_{-\infty}^{\infty} \phi(ax')\delta(x'-\xi')\,adx' \\
&= a\phi(\xi), \qquad (1.13)
\end{aligned}
$$

where we used $x' = x/a$, $\xi' = \xi/a$, $a > 0$.

1.2.1 Macaulay Brackets

A simplified notation to represent integrals of the δ function was introduced in the context of structural mechanics by Macaulay. In this notation

$$\delta(x-\xi) = \langle x-\xi \rangle^{-1}, \qquad (1.14)$$

$$h(x-\xi) = \langle x-\xi \rangle^{0}, \qquad (1.15)$$

$$\int \langle x-\xi \rangle^{n} dx = \frac{1}{n+1}\langle x-\xi \rangle^{n+1}, \quad n \neq -1, \qquad (1.16)$$

$$\int \langle x-\xi \rangle^{-1} dx = \langle x-\xi \rangle^{0}. \qquad (1.17)$$

All of these functions are zero when the quantity inside the brackets is negative. For $n < 0$, some books omit the factor $1/(n+1)$ in the integral. We may include higher derivatives of the delta function in this group. In one-dimensional problems, such as the deflection of beams under concentrated loads, this notation is useful.

1.2.2 Higher Dimensions

In an n-dimensional Euclidian space $\mathbf{R}^n$ with coordinates $(x_1, x_2, \ldots, x_n)$, we use the simplified notation for the infinitesimal volume,

$$dx_1 dx_2 \ldots dx_n = d\boldsymbol{x}, \tag{1.18}$$

and the same for functions

$$\phi(x_1, x_2, \ldots, x_n) = \phi(\boldsymbol{x}), \quad \delta(x_1, x_2, \ldots, x_n) = \delta(\boldsymbol{x}). \tag{1.19}$$

Then the n-dimensional integral,

$$\int_{\mathbf{R}^n} \phi(\boldsymbol{x})\delta(\boldsymbol{x})\, d\boldsymbol{x} = \phi(\mathbf{0}). \tag{1.20}$$

More often we encounter situations involving two and three-dimensional spaces and cartesian coordinates (x, y) or (x, y, z), and the above result directly applies. When we use polar coordinates (or spherical coordinates) the appropriate area element (or volume element)

$$dA = r\, dr\, d\theta \quad (\text{or} \quad dV = r^2 \sin^2\phi\, dr\, d\phi\, d\theta) \tag{1.21}$$

is used. For example,

$$\int_0^\infty \int_0^{2\pi} f(r,\theta)\delta(r - r_0, \theta - \theta_0) r\, dr\, d\theta = f(r_0, \theta_0) \tag{1.22}$$

1.2.3 Test Functions, Linear Functionals, and Distributions

We conclude this section by introducing the idea of generalized functions or distributions as linear functionals over test functions.

A function, $\phi(x)$, is called a test function if (a) $\phi \in \mathcal{C}^\infty$, (b) it has a closed bounded (compact) support, and (c) ϕ and all of its derivatives decrease to zero faster than any power of $|x|^{-1}$.

A linear functional $\mathcal{T}$ of ϕ maps it into a scalar. This is done using an integral over $-\infty$ to ∞ as an inner product with some other sequence or distribution, f. If we denote this mapping as

$$\mathcal{T}_f[\phi] = \int_{-\infty}^{\infty} f(x)\phi(x)\,dx, \tag{1.23}$$

then the δ-distribution is defined by the relation

$$\mathcal{T}_\delta[\phi] = \int_{-\infty}^{\infty} \delta(x)\phi(x)\,dx = \phi(0). \tag{1.24}$$

A sequence $\delta_n (n = 0, 1, \ldots, \infty)$ converges to the δ-function if

$$\lim_{n\to\infty} \mathcal{T}_{\delta_n}[\phi] \to \phi(0). \tag{1.25}$$

A distribution $\mu(x)$ is the derivative of the δ-distribution if

$$\mathcal{T}_\mu[\phi] = -\phi'(0), \tag{1.26}$$

as

$$\int_{-\infty}^{\infty} \delta'(x)\phi(x)\,dx = -\int_{-\infty}^{\infty} \delta(x)\phi'(x)\,dx = -\phi'(0). \tag{1.27}$$

This way, we can define higher-order derivatives of the delta function. In engineering, concentrated forces, charges, fluid flow sources, vortex lines, and the like are represented using delta functions. The delta function is also called a unit impulse function in control theory.

1.2.4 Examples: Delta Function

Using the property, for any test function ϕ,

$$\int_{-\infty}^{\infty} \phi(x)\psi(x)\,dx = \phi(\xi) \tag{1.28}$$

implies

$$\psi(x) = \delta(x - \xi), \tag{1.29}$$

prove that, for $\alpha, \beta \neq 0$,

(a)

$$\frac{\partial}{\partial\alpha}\delta(\alpha x) = -\frac{1}{\alpha^2}\delta(x), \tag{1.30}$$

(b)

$$\delta(e^{\alpha x}-\beta)=\frac{1}{\alpha\beta}\delta\left(x-\frac{\ln\beta}{\alpha}\right). \tag{1.31}$$

We may solve these examples using the basic properties of the delta function as follows:

(a)

$$\begin{aligned}\int_{-\infty}^{\infty}\phi(x)\frac{\partial}{\partial\alpha}\delta(\alpha x)\,dx &= \frac{\partial}{\partial\alpha}\int_{-\infty}^{\infty}\phi(x)\delta(\alpha x)\,dx\\ &= \frac{\partial}{\partial\alpha}\int_{-\infty}^{\infty}\phi(x/\alpha)\delta(x)\,dx/\alpha\\ &= \frac{\partial}{\partial\alpha}\frac{1}{\alpha}\phi(0)\\ &= -\frac{1}{\alpha^2}\phi(0). \end{aligned}\tag{1.32}$$

Comparing

$$\frac{\partial}{\partial\alpha}\delta(\alpha x)=-\frac{1}{\alpha^2}\delta(x). \tag{1.33}$$

(b)

$$\begin{aligned}\int_{-\infty}^{\infty}\phi(x)\delta(e^{\alpha x}-\beta)\,dx &= \int_{-\infty}^{\infty}\phi\left(\frac{\ln y}{\alpha}\right)\delta(y-\beta)\frac{dy}{\alpha y}\\ &= \int_{-\infty}^{\infty}\frac{1}{\alpha y}\phi\left(\frac{\ln y}{\alpha}\right)\delta(y-\beta)\,dy\\ &= \frac{1}{\alpha\beta}\phi\left(\frac{\ln\beta}{\alpha}\right). \end{aligned}\tag{1.34}$$

Thus,

$$\delta(e^{\alpha x}-\beta)=\frac{1}{\alpha\beta}\delta\left(x-\frac{\ln\beta}{\alpha}\right). \tag{1.35}$$

Other well-known examples of delta sequences are

$$\delta_a=\frac{1}{\pi}\frac{a}{a^2+x^2},\quad \text{limit}\quad a\to 0, \tag{1.36}$$

$$\delta_\lambda=\frac{1}{\pi}\frac{\sin\lambda x}{x},\quad \text{limit}\quad \lambda\to\infty. \tag{1.37}$$

1.3 LINEAR DIFFERENTIAL OPERATORS

Consider the differential equation

$$Lu(x) = f(x); \quad a < x < b. \tag{1.38}$$

Here, $u(x)$ is the unknown, $f(x)$ is a given forcing function, and L is a differential operator. For a differential equation of order n, we need n boundary conditions. For the time being, let us assume all the needed boundary conditions are homogeneous. The differential operator L has the form

$$L = a_n(x)\frac{d^n}{dx^n} + a_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + a_0(x). \tag{1.39}$$

A linear operator satisfies the properties

$$L(u_1 + u_2) = Lu_1 + Lu_2, \tag{1.40}$$

$$L(cu) = cLu, \tag{1.41}$$

where u_1, u_2, and u are functions in $\mathcal{C}^n$, and c is a constant. Recall $\mathcal{C}^n$ indicates the set of differentiable functions with all derivatives up to and including the nth continuous.

1.3.1 Example: Boundary Conditions

For the system

$$\frac{d^2u}{dx^2} + u = \sin x; \quad u(1) = 1, \quad u(2) = 3, \tag{1.42}$$

with nonhomogeneous boundary conditions, we introduce a new dependent variable v, as

$$u = v + Ax + B. \tag{1.43}$$

The boundary conditions become

$$u(1) = 1 = v(1) + A + B, \tag{1.44}$$

$$u(2) = 3 = v(2) + 2A + B. \tag{1.45}$$