THEORY OF FUSION SYSTEMS

Fusion systems are a recent development in finite group theory and sit at the intersection of algebra and topology. This book is the first to deal comprehensively with this new and expanding field, taking the reader from the basics of the theory right to the state of the art.

Three motivational chapters, indicating the interaction of fusion and fusion systems in group theory, representation theory, and topology are followed by six chapters that explore the theory of fusion systems themselves. Starting with the basic definitions, the topics covered include: weakly normal and normal subsystems; morphisms and quotients; saturation theorems; results about control of fusion; and the local theory of fusion systems. At the end, there is also a discussion of exotic fusion systems.

Designed for use as a text and reference work, this book is suitable for graduate students and experts alike.

David A. Craven is a Junior Research Fellow in the Mathematical Institute at the University of Oxford.
CAMBRIDGE STUDIES IN ADVANCED MATHEMATICS

Editorial Board:
B. Bollobás, W. Fulton, A. Katok, F. Kirwan, P. Sarnak, B. Simon, B. Totaro

All the titles listed below can be obtained from good booksellers or from Cambridge University Press.

For a complete series listing visit: http://www.cambridge.org/mathematics

Already published

85 J. Carlson, S. Müller-Stach & C. Peters Period mappings and period domains
86 J. J. Duistermaat & J. A. C. Kolk Multidimensional real analysis, I
87 J. J. Duistermaat & J. A. C. Kolk Multidimensional real analysis, II
89 M. C. Golumbic & A. N. Trenk Tolerance graphs
90 L. H. Harper Global methods for combinatorial isoperimetric problems
91 I. Moerdijk & J. Mrčun Introduction to foliations and Lie groupoids
92 J. Kollár, K. E. Smith & A. Corti Rational and nearly rational varieties
93 D. Applebaum Lévy processes and stochastic calculus (1st Edition)
94 B. Conrad Modular forms and the Ramanujan conjecture
95 M. Schechter An introduction to nonlinear analysis
96 R. Carter Lie algebras of finite and affine type
97 H. L. Montgomery & R. C. Vaughan Multiplicative number theory, I
98 I. Chavel Riemannian geometry (2nd Edition)
99 D. Goldfeld Automorphic forms and L-functions for the group GL(n,R)
100 M. B. Marcus & J. Rosen Markov processes, Gaussian processes, and local times
101 P. Gille & T. Szamuely Central simple algebras and Galois cohomology
102 J. Bertoin Random fragmentation and coagulation processes
103 E. Frenkel Langlands correspondence for loop groups
104 A. Ambrosetti & A. Malchiodi Nonlinear analysis and semilinear elliptic problems
105 T. Tao & V. H. Vu Additive combinatorics
106 E. B. Davies Linear operators and their spectra
107 K. Kodaira Complex analysis
108 T. Ceccherini-Silberstein, F. Scarabotti & F. Tolli Harmonic analysis on finite groups
109 H. Geiges An introduction to contact topology
110 J. Faraut Analysis on Lie groups: An introduction
111 E. Park Complex topological K-theory
112 D. W. Stroock Partial differential equations for probabilists
113 A. Kirillov, Jr An introduction to Lie groups and Lie algebras
114 F. Gesztesy et al. Soliton equations and their algebro-geometric solutions, II
115 E. de Faria & W. de Melo Mathematical tools for one-dimensional dynamics
116 D. Applebaum Lévy processes and stochastic calculus (2nd Edition)
117 T. Szamuel Galois groups and fundamental groups
118 G. W. Anderson, A. Guionnet & O. Zeitouni An introduction to random matrices
119 C. Perez-Garcia & W. H. Schikhof Locally convex spaces over non-Archimedean valued fields
120 P. K. Friz & N. B. Victoir Multidimensional stochastic processes as rough paths
121 T. Ceccherini-Silberstein, F. Scarabotti & F. Tolli Representation theory of the symmetric groups
122 S. Kalikow & R. McCutcheon An outline of ergodic theory
123 G. F. Lawler & V. Limic Random walk: A modern introduction
124 K. Lux & H. Pahlings Representations of groups
125 K. S. Kedlaya p-adic differential equations
126 R. Beals & R. Wong Special functions
127 E. de Faria & W. de Melo Mathematical aspects of quantum field theory
128 A. Terras Zeta functions of graphs
129 D. Goldfeld & J. Hundley Automorphic representations and L-functions for the general linear group, I
130 D. Goldfeld & J. Hundley Automorphic representations and L-functions for the general linear group, II
131 D. A. Craven The theory of fusion systems
132 J. Vaananen Models and games
133 G. Malle & D. Testerman Linear algebraic groups and finite groups of Lie type
The Theory of Fusion Systems

An Algebraic Approach

D A V I D  A .  C R A V E N

University of Oxford
# Contents

**Preface**

**PART I MOTIVATION**

1 Fusion in finite groups
   1.1 Control of fusion  
   1.2 Normal $p$-complements  
   1.3 Alperin’s fusion theorem  
   1.4 The focal subgroup theorem  
   1.5 Fusion systems  
   Exercises

2 Fusion in representation theory
   2.1 Blocks of finite groups  
   2.2 The Brauer morphism and relative traces  
   2.3 Brauer pairs  
   2.4 Defect groups and the first main theorem  
   2.5 Fusion systems of blocks  
   Exercises

3 Fusion in topology
   3.1 Simplicial sets  
   3.2 Classifying spaces  
   3.3 Simplicial and cosimplicial objects  
   3.4 Bousfield–Kan completions  
   3.5 The centric linking systems of groups  
   3.6 Constrained fusion systems  
   Exercises
## Contents

### PART II THE THEORY

4 Fusion systems 93

4.1 Saturated fusion systems 94
4.2 Normalizing and centralizing 101
4.3 The equivalent definitions 105
4.4 Local subsystems 108
4.5 Centric and radical subgroups 117
4.6 Alperin’s fusion theorem 121
4.7 Weak and strong closure 127
Exercises 131

5 Weakly normal subsystems, quotients, and morphisms 134

5.1 Morphisms of fusion systems 135
5.2 The isomorphism theorems 141
5.3 Normal subgroups 148
5.4 Weakly normal subsystems 150
5.5 Correspondences for quotients 160
5.6 Simple fusion systems 171
5.7 Soluble fusion systems 181
Exercises 186

6 Proving saturation 188

6.1 The surjectivity property 189
6.2 Reduction to centric subgroups 193
6.3 Invariant maps 201
6.4 Weakly normal maps 205
Exercises 212

7 Control in fusion systems 215

7.1 Resistance 216
7.2 Glauberman functors 221
7.3 The $ZJ$-theorems 227
7.4 Normal $p$-complement theorems 232
7.5 The hyperfocal and residual subsystems 236
7.6 Bisets 252
7.7 The transfer 260
Exercises 267

8 Local theory of fusion systems 270

8.1 Normal subsystems 271
8.2 Weakly normal and normal subsystems 275
Contents

8.3 Intersections of subsystems 280
8.4 Constraint and normal subsystems 287
8.5 Central products 295
8.6 The generalized Fitting subsystem 299
8.7 $L$-balance 308
Exercises 314

9 Exotic fusion systems 317
9.1 Extraspecial $p$-groups 318
9.2 The Solomon fusion system 326
9.3 Blocks of finite groups 330
9.4 Block exotic fusion systems 335
9.5 Abstract centric linking systems 343
9.6 Higher limits and centric linking systems 349
Exercises 356

References 358
Index of notation 364
Index 366
Preface

It is difficult to pinpoint the origins of the theory of fusion systems: it could be argued that they stretch back to Burnside and Frobenius, with arguments about the fusion of \( p \)-elements of finite groups. Another viewpoint is that it really started with the theorems on fusion in finite groups, such as Alperin’s fusion theorem, or Grün’s theorems.

We will take as the starting point the important paper of Solomon [Sol74], which proves that, for a Sylow 2-subgroup \( P \) of Spin\(_7\)(3), there is a particular pattern of the fusion of involutions in \( P \) that, while not internally inconsistent, is not consistent with living inside a finite group. This is the first instance where the fusion of \( p \)-elements looks fine on its own, but is incompatible with coming from a finite group.

Unpublished work of Puig during the 1990s and even before (some of which is collected in [Pui06]), together with work of Alperin–Broué [AB79], is the basis for constructing a fusion system for a \( p \)-block of a finite group. It was with Puig’s work where the axiomatic foundations of fusion systems started, and where some of the fundamental notions begin. It cannot be overestimated how much the current theory of fusion systems owes to Puig, both in originating the definition and related notions, and in furthering the theory.

Various results that could be considered part of local finite group theory (the study of \( p \)-subgroups, normalizers, conjugacy, and so on) were extended to \( p \)-blocks of finite groups during the 1990s and early part of the twenty-first century, but at the time were not viewed as taking place in the more general setting of fusion systems. With this theory now becoming more popular, more and more results are being cast in this language, and extended to this area.

The internal theory of fusion systems, starting from these foundations, has developed rapidly, and in many respects has mimicked the theory of
Preface

finite groups, with normal subsystems, quotients, the generalized Fitting subsystem, composition series, soluble fusion systems, and so on. However, there is also a topological aspect to this theory.

Along with the representation theory, topology has played an important role in the development of the theory: Benson [Ben98a] constructed a topological space that should be the 2-completed classifying space of a finite group whose fusion pattern matched that which Solomon considered. Since such a group does not exist, this space can be thought of as the shadow cast by an invisible group. Benson predicted that this topological space is but one facet of a general theory, a prediction that was confirmed with the development of $p$-local finite groups.

Although we will not cover the topic of $p$-local finite groups here (we only meet the definition in Chapter 9), they can be thought of as some data describing a $p$-completed classifying space of a fusion system. In the case where the fusion system arises from a finite group, the corresponding $p$-local finite group describes the normal $p$-completed classifying space.

In this direction, we have Oliver’s proof [Oli04] [Oli06] of the Martino–Priddy conjecture [MP96], which states that two finite groups have homotopy equivalent $p$-completed classifying spaces if and only if the fusion systems are isomorphic. The topological considerations have fuelled development in the algebraic aspects of fusion systems and vice versa, and the two viewpoints are somewhat intertwined. Having said that, we will not deal with the topological theory here beyond that which is given in Part I, and concentrate on the more algebraic aspects.

As this is a young subject, still in development, the foundations of the theory have not yet been solidified; indeed, there is some debate as to the correct definition of a fusion system! (It should be noted that the definitions are all equivalent, and so the choice is only apparent.) The definition of a ‘normal’ subsystem is also under discussion, and which definition is used often indicates the intended applications of the theory. Here we have made a choice based upon the evidence available now; this might change as time goes on. Since group theorists, representation theorists, and topologists all converge on this area, there are several different conventions and styles, as well as approaches.

The first three chapters are preliminary in nature, and deal with group theory, representation theory, and topology. The first chapter is essentially a run-through of the theory of fusion in finite groups, giv-
Preface

ing for example the $p$-complement theorems of Frobenius, Burnside, and Glauberman–Thompson. The second chapter introduces the representation theory aspects, and in particular develops the block theory needed to construct the fusion system associated to a $p$-block of a finite group. The third chapter develops the topological methods used in the theory, but since the main thrust of this work is the algebraic theory of fusion systems, we necessarily skip over many of the details in this chapter.

The remaining six chapters deal with the theory of fusion systems. The fourth chapter starts by defining fusion systems and in particular saturated fusion systems, then constructing local subsystems, proving Alperin’s fusion theorem, and introducing strongly closed subgroups. In Chapter 5, we start looking at the normal and quotient structure of fusion systems, introducing morphisms, quotients, weakly normal and characteristic subsystems, the centre, and so on. The sixth chapter deals with methods used to prove saturation, and introduces weakly normal maps as well.

Chapter 7 deals with topics around control of fusion, with analogues of the Glauberman–Thompson normal $p$-complement theorem, Glauberman’s $ZJ$-theorem, and the two normal subgroups $O^p(G)$ and $O^{p'}(G)$, the former of which is the object of the theory of transfer. After proving the existence of a certain kind of biset associated to any saturated fusion system in Section 7.6, we use the biset to develop the transfer for a fusion system.

Chapter 8 focuses on work of Aschbacher which attempts to translate some aspects of local finite group theory into the domain of fusion systems. We prove here that, for constrained fusion systems, there is a one-to-one correspondence between the normal subsystems of the fusion system and the normal subgroups of the associated model. Other highlights include a description of the generalized Fitting subsystem of a fusion system, and the proof of $L$-balance for fusion systems, which is considerably easier than the proof of the corresponding theorem for finite groups.

The final chapter consists of questions about exotic fusion systems (i.e., fusion systems that do not come from groups), with a few details on some of the known exotic fusion systems, theorems on which exotic fusion systems do not come from blocks of finite groups, and Oliver’s conjecture relating modular representation theory of $p$-groups to the existence and uniqueness of centric linking systems. A solution to this
conjecture would remove the requirement of the classification of the finite simple groups for the proof of the Martino–Priddy conjecture.

The choice of definitions and conventions has been influenced by the background of the author: as I am a group theorist and group representation theorist, the conventions here will be the standard group theory conventions, rather than topology conventions. In particular, homomorphisms will be composed from left to right. The only chapter where this will be relaxed is Chapter 3, the topological chapter; the reason for this is that to keep left-to-right notation would go against every other topology book in existence, and require writing functors on the right, something that I, even as a group theorist, cannot bring myself to do.

It remains for me to thank various people, most notably Adam Glesser for reading much of this work and for being a sounding board for various ideas, mathematical, pedagogical and notational. Thank you to George Raptis for reading Chapter 3, and explaining some of the topological ideas to an algebraist, making the exposition in that chapter considerably clearer. Proof reading and valuable comments were given by (in alphabetical order): Tobias Barthel, Michael Collins, Radha Kessar, Bob Oliver, Oscar Randal-Williams, George Raptis, Raphaël Rouquier, Jason Semeraro and Matt Towers. Any errors that remain in this work are, of course, my own.

David A. Craven, Oxford