1

Mathematical background

1.1 Cartesian and spherical coordinates

Two systems of orthogonal coordinates are used in this book, sometimes interchangeably. Cartesian coordinates (x, y, z) are used for a system with rectangular geometry, and spherical polar coordinates (r, θ, ϕ) are used for spherical geometry. The relationship between these reference systems is shown in Fig. 1.1(a). The convention used here for spherical geometry is defined as follows: the radial distance from the origin of the coordinates is denoted r, the polar angle θ (geographic equivalent: the co-latitude) lies between the radius and the z-axis (geographic equivalent: Earth's rotation axis), and the azimuthal angle ϕ in the x–y plane is measured from the x-axis (geographic equivalent: longitude). Position on the surface of a sphere (constant r) is described by the two angles θ and ϕ. The Cartesian and spherical polar coordinates are linked as illustrated in Fig. 1.1(b) by the relationships

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \end{aligned} \tag{1.1}$$

1.2 Complex numbers

The numbers we most commonly use in daily life are *real* numbers. Some of them are also *rational* numbers. This means that they can be expressed as the quotient of two integers, with the condition that the denominator of the quotient must not equal zero. When the denominator is 1, the real number is an integer. Thus 4, 4/5, 123/456 are all rational numbers. A real number can also be *irrational*, which means it cannot be expressed as the quotient of two integers.

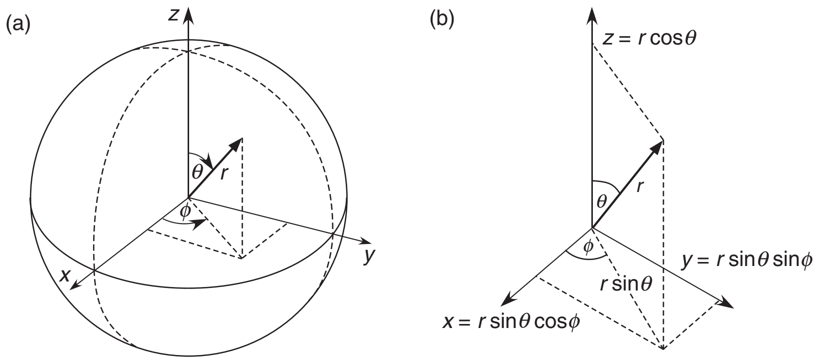

Fig. 1.1. (a) Cartesian and spherical polar reference systems. (b) Relationships between the Cartesian and spherical polar coordinates.

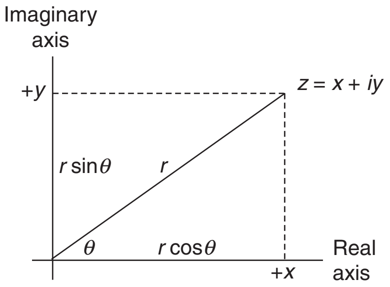

Fig. 1.2. Representation of a complex number on an Argand diagram.

Familiar examples are π, e (the base of natural logarithms), and some square roots, such as $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, etc. The irrational numbers are real numbers that do not terminate or repeat when expressed as decimals.

In certain analyses, such as determining the roots of an equation, it is necessary to find the square root of a negative real number, e.g. $\sqrt{(-y^2)}$, where y is real. The result is an *imaginary* number. The negative real number can be written as $(-1)y^2$, and its square root is then $\sqrt{(-1)}y$. The quantity $\sqrt{(-1)}$ is written i and is known as the imaginary unit, so that $\sqrt{(-y^2)}$ becomes $\pm iy$.

A complex number comprises a real part and an imaginary part. For example, $z = x + iy$, in which x and y are both real numbers, is a complex number with a *real part* x and an *imaginary part* y. The composition of a complex number can be illustrated graphically with the aid of the *complex plane* (Fig. 1.2). The real part is plotted on the horizontal axis, and the imaginary part on the vertical axis. The two independent parts are orthogonal on the plot and the complex number z

is represented by their vector sum, defining a point on the plane. The distance r of the point from the origin is given by

$$r = \sqrt{x^2 + y^2} \tag{1.2}$$

The line joining the point to the origin makes an angle θ with the real (x-)axis, and so r has real and imaginary components $r\cos\theta$ and $r\sin\theta$, respectively. The complex number z can be written in polar form as

$$z = r(\cos\theta + i\sin\theta) \tag{1.3}$$

It is often useful to write a complex number in the exponential form introduced by Leonhard Euler in the late eighteenth century. To illustrate this we make use of infinite power series; this topic is described in Section 1.10. The exponential function, $\exp(x)$, of a variable x can be expressed as a power series as in (1.135). On substituting $x = i\theta$, the power series becomes

$$\begin{aligned} \exp(i\theta) &= 1 + (i\theta) + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \cdots \\ &= 1 + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^6}{6!} \cdots + (i\theta) + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^5}{5!} + \cdots \\ &= \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots\right) \end{aligned} \tag{1.4}$$

Comparison with (1.135) shows that the first bracketed expression on the right is the power series for $\cos\theta$; the second is the power series for $\sin\theta$. Therefore

$$\exp(i\theta) = \cos\theta + i\sin\theta \tag{1.5}$$

On inserting (1.5) into (1.3), the complex number z can be written in exponential form as

$$z = r\exp(i\theta) \tag{1.6}$$

The quantity r is the *modulus* of the complex number and θ is its *phase*.

Conversely, using (1.5) the cosine and sine functions can be defined as the sum or difference of the complex exponentials $\exp(i\theta)$ and $\exp(-i\theta)$:

$$\begin{aligned} \cos\theta &= \frac{\exp(i\theta) + \exp(-i\theta)}{2} \\ \sin\theta &= \frac{\exp(i\theta) - \exp(-i\theta)}{2i} \end{aligned} \tag{1.7}$$

1.3 Vector relationships

A scalar quantity is characterized only by its magnitude; a vector has both magnitude and direction; a unit vector has unit magnitude and the direction of the quantity it represents. In this overview the unit vectors for Cartesian coordinates (x, y, z) are written $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$; unit vectors in spherical polar coordinates (r, θ, ϕ) are denoted $(\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi)$. The unit vector normal to a surface is simply denoted $\mathbf{n}$.

1.3.1 Scalar and vector products

The scalar product of two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as the product of their magnitudes and the cosine of the angle α between the vectors:

$$\mathbf{a} \cdot \mathbf{b} = ab \cos \alpha \tag{1.8}$$

If the vectors are orthogonal, the cosine of the angle α is zero and

$$\mathbf{a} \cdot \mathbf{b} = 0 \tag{1.9}$$

The vector product of two vectors is another vector, whose direction is perpendicular to both vectors, such that a right-handed rule is observed. The magnitude of the vector product is the product of the individual vector magnitudes and the sine of the angle α between the vectors:

$$|\mathbf{a} \times \mathbf{b}| = ab \sin \alpha \tag{1.10}$$

If $\mathbf{a}$ and $\mathbf{b}$ are parallel, the sine of the angle between them is zero and

$$\mathbf{a} \times \mathbf{b} = 0 \tag{1.11}$$

Applying these rules to the unit vectors $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$, which are normal to each other and have unit magnitude, it follows that their scalar products are

$$\begin{aligned} \mathbf{e}_x \cdot \mathbf{e}_y = \mathbf{e}_y \cdot \mathbf{e}_z = \mathbf{e}_z \cdot \mathbf{e}_x = 0 \\ \mathbf{e}_x \cdot \mathbf{e}_x = \mathbf{e}_y \cdot \mathbf{e}_y = \mathbf{e}_z \cdot \mathbf{e}_z = 1 \end{aligned} \tag{1.12}$$

The vector products of the unit vectors are

$$\begin{aligned} \mathbf{e}_x \times \mathbf{e}_y &= \mathbf{e}_z \\ \mathbf{e}_y \times \mathbf{e}_z &= \mathbf{e}_x \\ \mathbf{e}_z \times \mathbf{e}_x &= \mathbf{e}_y \\ \mathbf{e}_x \times \mathbf{e}_x &= \mathbf{e}_y \times \mathbf{e}_y = \mathbf{e}_z \times \mathbf{e}_z = 0 \end{aligned} \tag{1.13}$$

A vector **a** with components (a_x, a_y, a_z) is expressed in terms of the unit vectors ($\mathbf{e}_x$, $\mathbf{e}_y$, $\mathbf{e}_z$) as

$$\mathbf{a} = a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z \tag{1.14}$$

The scalar product of the vectors **a** and **b** is found by applying the relationships in (1.12):

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= \left(a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z\right)\cdot\left(b_x\mathbf{e}_x + b_y\mathbf{e}_y + b_z\mathbf{e}_z\right)\\ &= a_xb_x + a_yb_y + a_zb_z\end{aligned} \tag{1.15}$$

The vector product of the vectors **a** and **b** is found by using (1.13):

$$\begin{aligned}\mathbf{a}\times\mathbf{b} &= \left(a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z\right)\times\left(b_x\mathbf{e}_x + b_y\mathbf{e}_y + b_z\mathbf{e}_z\right)\\ &= \left(a_yb_z - a_zb_y\right)\mathbf{e}_x + \left(a_zb_x - a_xb_z\right)\mathbf{e}_y + \left(a_xb_y - a_yb_x\right)\mathbf{e}_z\end{aligned} \tag{1.16}$$

This result leads to a convenient way of evaluating the vector product of two vectors, by writing their components as the elements of a determinant, as follows:

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix}\mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z\\ a_x & a_y & a_z\\ b_x & b_y & b_z\end{vmatrix} \tag{1.17}$$

The following relationships may be established, in a similar manner to the above, for combinations of scalar and vector products of the vectors **a**, **b**, and **c**:

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b}) \tag{1.18}$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{b}(\mathbf{c}\cdot\mathbf{a}) - \mathbf{c}(\mathbf{a}\cdot\mathbf{b}) \tag{1.19}$$

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \mathbf{b}(\mathbf{c}\cdot\mathbf{a}) - \mathbf{a}(\mathbf{b}\cdot\mathbf{c}) \tag{1.20}$$

1.3.2 Vector differential operations

The vector differential operator ∇ is defined relative to Cartesian axes (x, y, z) as

$$\nabla = \mathbf{e}_x\frac{\partial}{\partial x} + \mathbf{e}_y\frac{\partial}{\partial y} + \mathbf{e}_z\frac{\partial}{\partial z} \tag{1.21}$$

The vector operator ∇ determines the gradient of a scalar function, which may be understood as the rate of change of the function in the direction of each of the reference axes. For example, the gradient of the scalar function φ with respect to Cartesian axes is the vector

$$\nabla\varphi = \mathbf{e}_x\frac{\partial\varphi}{\partial x} + \mathbf{e}_y\frac{\partial\varphi}{\partial y} + \mathbf{e}_z\frac{\partial\varphi}{\partial z} \tag{1.22}$$

The vector operator ∇ can operate on either a scalar quantity or a vector. The scalar product of ∇ with a vector is called the *divergence* of the vector. Applied to the vector **a** it is equal to

$$\begin{aligned}\nabla\cdot\mathbf{a} &= \left(\mathbf{e}_x\frac{\partial}{\partial x} + \mathbf{e}_y\frac{\partial}{\partial y} + \mathbf{e}_z\frac{\partial}{\partial z}\right)\cdot(a_x\mathbf{e}_x + a_y\mathbf{e}_y + a_z\mathbf{e}_z)\\ &= \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}\end{aligned} \tag{1.23}$$

If the vector **a** is defined as the gradient of a scalar potential φ, as in (1.22), we can substitute potential gradients for the vector components (a_x, a_y, a_z). This gives

$$\nabla\cdot\nabla\varphi = \frac{\partial}{\partial x}\left(\frac{\partial\varphi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial\varphi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial\varphi}{\partial z}\right) \tag{1.24}$$

By convention the scalar product $(\nabla\cdot\nabla)$ on the left is written ∇^2. The resulting identity is very important in potential theory and is encountered frequently. In Cartesian coordinates it is

$$\nabla^2\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} \tag{1.25}$$

The vector product of ∇ with a vector is called the *curl* of the vector. The curl of the vector **a** may be obtained using a determinant similar to (1.17):

$$\nabla\times\mathbf{a} = \begin{vmatrix}\mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ a_x & a_y & a_z\end{vmatrix} \tag{1.26}$$

In expanded format, this becomes

$$\nabla\times\mathbf{a} = \left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right)\mathbf{e}_x + \left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right)\mathbf{e}_y + \left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)\mathbf{e}_z \tag{1.27}$$

The curl is sometimes called the *rotation* of a vector, because of its physical interpretation (Box 1.1). Some commonly encountered divergence and curl operations on combinations of the scalar quantity φ and the vectors **a** and **b** are listed below:

$$\nabla\cdot(\varphi\mathbf{a}) = (\nabla\varphi)\cdot\mathbf{a} + \varphi(\nabla\cdot\mathbf{a}) \tag{1.28}$$

Box 1.1. The curl of a vector

The curl of a vector at a given point is related to the circulation of the vector about that point. This interpretation is best illustrated by an example, in which a fluid is rotating about a point with constant angular velocity $\boldsymbol{\omega}$. At distance $\mathbf{r}$ from the point the linear velocity of the fluid $\mathbf{v}$ is equal to $\boldsymbol{\omega} \times \mathbf{r}$. Taking the curl of $\mathbf{v}$, and applying the identity (1.31) with $\boldsymbol{\omega}$ constant,

$$\nabla \times \mathbf{v} = \nabla \times (\boldsymbol{\omega} \times \mathbf{r}) = \boldsymbol{\omega}(\nabla \cdot \mathbf{r}) - (\boldsymbol{\omega} \cdot \nabla)\mathbf{r} \tag{1}$$

To evaluate the first term on the right, we use rectangular coordinates (x, y, z):

$$\begin{aligned}\boldsymbol{\omega}(\nabla \cdot \mathbf{r}) &= \boldsymbol{\omega}\left(\mathbf{e}_x \frac{\partial}{\partial x} + \mathbf{e}_y \frac{\partial}{\partial y} + \mathbf{e}_z \frac{\partial}{\partial z}\right) \cdot (x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z) \\ &= \boldsymbol{\omega}(\mathbf{e}_x \cdot \mathbf{e}_x + \mathbf{e}_y \cdot \mathbf{e}_y + \mathbf{e}_z \cdot \mathbf{e}_z) = 3\boldsymbol{\omega}\end{aligned} \tag{2}$$

The second term is

$$\begin{aligned}(\boldsymbol{\omega} \cdot \nabla)r &= \left(\omega_x \frac{\partial}{\partial x} + \omega_y \frac{\partial}{\partial y} + \omega_z \frac{\partial}{\partial z}\right) \cdot (x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z) \\ &= \omega_x \mathbf{e}_x + \omega_y \mathbf{e}_y + \omega_z \mathbf{e}_z = \boldsymbol{\omega}\end{aligned} \tag{3}$$

Combining the results gives

$$\nabla \times \mathbf{v} = 2\boldsymbol{\omega} \tag{4}$$

$$\boldsymbol{\omega} = \frac{1}{2}(\nabla \times \mathbf{v}) \tag{5}$$

Because of this relationship between the angular velocity and the linear velocity of a fluid, the curl operation is often interpreted as the *rotation* of the fluid. When $\nabla \times \mathbf{v} = 0$ everywhere, there is no rotation. A vector that satisfies this condition is said to be *irrotational*.

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b}) \tag{1.29}$$

$$\nabla \times (\varphi \mathbf{a}) = (\nabla \varphi) \times \mathbf{a} + \varphi(\nabla \times \mathbf{a}) \tag{1.30}$$

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = \mathbf{a}(\nabla \cdot \mathbf{b}) - \mathbf{b}(\nabla \cdot \mathbf{a}) - (\mathbf{a} \cdot \nabla)\mathbf{b} + (\mathbf{b} \cdot \nabla)\mathbf{a} \tag{1.31}$$

$$\nabla \times (\nabla \varphi) = 0 \tag{1.32}$$

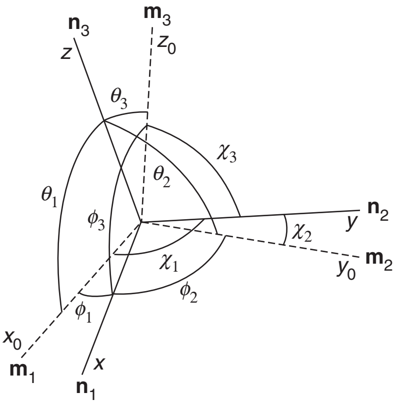

Fig. 1.3. Two sets of Cartesian coordinate axes, (x, y, z) and (x_0, y_0, z_0), with corresponding unit vectors $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3)$ and $(\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3)$, rotated relative to each other.

$$\nabla \cdot (\nabla \times \mathbf{a}) = 0 \tag{1.33}$$

$$\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a} \tag{1.34}$$

It is a worthwhile exercise to establish these identities from basic principles, especially (1.19) and (1.31)–(1.34), which will be used in later chapters.

1.4 Matrices and tensors

1.4.1 The rotation matrix

Consider two sets of orthogonal Cartesian coordinate axes (x, y, z) and (x_0, y_0, z_0) that are inclined to each other as in Fig. 1.3. The x_0-axis makes angles $(\phi_1, \chi_1, \theta_1)$ with each of the (x, y, z) axes in turn. Similar sets of angles $(\phi_2, \chi_2, \theta_2)$ and $(\phi_3, \chi_3, \theta_3)$ are defined by the orientations of the y_0- and z_0-axes, respectively, to the (x, y, z) axes. Let the unit vectors along the (x, y, z) and (x_0, y_0, z_0) axes be $(\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3)$ and $(\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3)$, respectively. The vector $\mathbf{r}$ can be expressed in either system, i.e., $\mathbf{r} = \mathbf{r}(x, y, z) = \mathbf{r}(x_0, y_0, z_0)$, or, in terms of the unit vectors,

$$\mathbf{r} = x\mathbf{n}_1 + y\mathbf{n}_2 + z\mathbf{n}_3 = x_0\mathbf{m}_1 + y_0\mathbf{m}_2 + z_0\mathbf{m}_3 \tag{1.35}$$

We can write the scalar product $(\mathbf{r} \cdot \mathbf{m}_1)$ as

$$\mathbf{r} \cdot \mathbf{m}_1 = x\mathbf{n}_1 \cdot \mathbf{m}_1 + y\mathbf{n}_2 \cdot \mathbf{m}_1 + z\mathbf{n}_3 \cdot \mathbf{m}_1 = x_0 \tag{1.36}$$

The scalar product $(\mathbf{n}_1 \cdot \mathbf{m}_1) = \cos\phi_1 = \alpha_{11}$ defines α_{11} as the *direction cosine* of the x_0-axis with respect to the x-axis (Box 1.2). Similarly, $(\mathbf{n}_2 \cdot \mathbf{m}_1) = \cos\chi_1 = \alpha_{12}$

and $(\mathbf{n}_3 \cdot \mathbf{m}_1) = \cos\theta_1 = \alpha_{13}$ define α_{12} and α_{13} as the direction cosines of the x_0-axis with respect to the y- and z-axes, respectively. Thus, (1.36) is equivalent to

$$x_0 = \alpha_{11}x + \alpha_{12}y + \alpha_{13}z \tag{1.37}$$

On treating the y_0- and z_0-axes in the same way, we get their relationships to the (x, y, z) axes:

$$\begin{aligned} y_0 &= \alpha_{21}x + \alpha_{22}y + \alpha_{23}z \\ z_0 &= \alpha_{31}x + \alpha_{32}y + \alpha_{33}z \end{aligned} \tag{1.38}$$

The three equations can be written as a single matrix equation

$$\begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = M \begin{bmatrix} x \\ y \\ z \end{bmatrix} \tag{1.39}$$

The coefficients α_{nm} (n = 1, 2, 3; m = 1, 2, 3) are the cosines of the interaxial angles. By definition, $\alpha_{12} = \alpha_{21}$, $\alpha_{23} = \alpha_{32}$, and $\alpha_{31} = \alpha_{13}$, so the square matrix M is symmetric. It transforms the components of the vector in the (x, y, z) coordinate system to corresponding values in the (x_0, y_0, z_0) coordinate system. It is thus equivalent to a rotation of the reference axes.

Because of the orthogonality of the reference axes, useful relationships exist between the direction cosines, as shown in Box 1.2. For example,

$$(\alpha_{11})^2 + (\alpha_{12})^2 + (\alpha_{13})^2 = \cos^2\phi_1 + \cos^2\chi_1 + \cos^2\theta_1 = \frac{1}{r^2}\left(x^2 + y^2 + z^2\right) = 1 \tag{1.40}$$

and

$$\alpha_{11}\alpha_{21} + \alpha_{12}\alpha_{22} + \alpha_{13}\alpha_{23} = \cos\phi_1\cos\phi_2 + \cos\chi_1\cos\chi_2 + \cos\theta_1\cos\theta_2 = 0 \tag{1.41}$$

The last summation is zero because it is the cosine of the right angle between the x_0-axis and the y_0-axis.

These two results can be summarized as

$$\sum_{k=1}^{3} \alpha_{mk}\alpha_{nk} = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases} \tag{1.42}$$

1.4.2 Eigenvalues and eigenvectors

The transpose of a matrix X with elements α_{nm} is a matrix with elements α_{mn} (i.e., the elements in the rows are interchanged with corresponding elements in

Box 1.2. Direction cosines

The vector $\mathbf{r}$ is inclined at angles α, β, and γ, respectively, to orthogonal reference axes (x, y, z) with corresponding unit vectors $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$, as in Fig. B1.2. The vector $\mathbf{r}$ can be written

$$\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z \tag{1}$$

where (x, y, z) are the components of $\mathbf{r}$ with respect to these axes. The scalar products of $\mathbf{r}$ with $\mathbf{e}_x$, $\mathbf{e}_y$, and $\mathbf{e}_z$ are

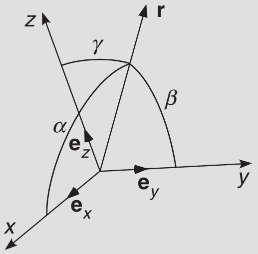

Fig. B1.2. Angles α, β, and γ define the tilt of a vector $\mathbf{r}$ relative to orthogonal reference axes (x, y, z), respectively. The unit vectors $(\mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z)$ define the coordinate system.

$$\begin{aligned} \mathbf{r} \cdot \mathbf{e}_x &= x = r \cos\alpha \\ \mathbf{r} \cdot \mathbf{e}_y &= y = r \cos\beta \\ \mathbf{r} \cdot \mathbf{e}_z &= z = r \cos\gamma \end{aligned} \tag{2}$$

Therefore, the vector $\mathbf{r}$ in (1) is equivalent to

$$\mathbf{r} = (r\cos\alpha)\mathbf{e}_x + (r\cos\beta)\mathbf{e}_y + (r\cos\gamma)\mathbf{e}_z \tag{3}$$

The unit vector $\mathbf{u}$ in the direction of $\mathbf{r}$ has the same direction as $\mathbf{r}$ but its magnitude is unity:

$$\mathbf{u} = \frac{\mathbf{r}}{r} = (\cos\alpha)\mathbf{e}_x + (\cos\beta)\mathbf{e}_y + (\cos\gamma)\mathbf{e}_z = l\mathbf{e}_x + m\mathbf{e}_y + n\mathbf{e}_z \tag{4}$$

where (l, m, n) are the cosines of the angles that the vector $\mathbf{r}$ makes with the reference axes, and are called the direction cosines of $\mathbf{r}$. They are useful for describing the orientations of lines and vectors.