Introduction

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No scientific theory has caused more puzzlement and confusion than quantum theory. Beginning in 1900, the theory developed in fits and starts and found a consistent mathematical framing only when John von Neumann published his *Mathematische Grundlagen der Quantenmechanik* in 1932 [12]. But even today, we struggle to understand the world as pictured by quantum theory. Physics is supposed to help us to *understand* the world, and yet quantum theory makes it seem a very strange place.

One might be tempted to push aside our puzzlement as the result of our clinging to a primitive worldview. But our puzzlement is not merely a psychological obstacle; it is also an obstacle to the development of physics itself. This obstacle is encountered primarily in our attempts to unify quantum theory and the general theory of relativity. As argued persuasively by Chris Isham (who is represented in this volume), Lee Smolin [10], and others, the primary obstacle between us and future physics is our own failure to understand the conceptual foundations of current physical theories.

How, then, are we to make conceptual progress? What is the process by which we find a new perspective, a perspective in which previously puzzling phenomena find a place in an intelligible—and perhaps beautiful—structure?

I do not wish to make prescriptions or to claim that conceptual progress can be achieved in only one way. But this book begins with the *Ansatz* that conceptual progress might be achieved through free creations of the human intellect. And where are we to find this free creative activity? According to a distinguished tradition, beginning with the philosopher Immanuel Kant and running through the philosopher-mathematicians Gottlob Frege and L. E. J. Brouwer, the mathematical sciences are in the business of constructing new and "fruitful" concepts. Thus, this book begins from the assumption that creative developments in mathematics might catalyze the conceptual advances that enable us to understand our current physical theories (in particular, quantum theory) and thereby to promote future advances in physics.

Because the guiding theme of this book is methodological rather than thematic, its chapters are naturally written from diverse perspectives, unified only by the attempt to introduce new concepts that will aid our understanding of current physics, as well as the growth of future physics. Some of the authors are mathematicians (Conway, de Groote, Kochen, Spitters); some mathematical physicists (Baez, Coecke, Döring, Heunen, Isham, Landsman, Lauda, Summers); some theoretical physicists (Brukner, Dakić, Hardy); and some philosophers (Bub, Redéi). But regardless of their professional affiliations, each author takes an interdisciplinary approach that combines methods and ideas from physics, mathematics, and philosophy. In the remainder of this Introduction, we briefly overview the various chapters and their contribution to the ongoing task of making sense of the physical world.

I.1 Beyond Hilbert Space

Quantum theory was born from a failure—namely, the failure of classical mechanics to provide accurate statistical predictions (e.g., in the case of blackbody radiation). Indeed, it was Einstein who saw clearly in the years between 1900 and 1905 that the framework of classical physics required a major overhaul. But unlike the theory of relativity, quantum theory did not result from a single stroke of genius. Rather, the following three decades witnessed a prolonged struggle by some of the century's greatest minds, including Niels Bohr, Arnold Sommerfeld, Max Born, Werner Heisenberg, Erwin Schrödinger, and Paul Dirac. Throughout this period, the developing "quantum" theory was not much more than a cobbled-together set of statistical rules of thumb that provided more accurate predictions than classical statistical mechanics.

In the second half of the 1920s, these struggles yielded two major mathematical advances: first, Schrödinger's introduction of the wave mechanical formalism; and second, Heisenberg's introduction of matrix mechanics. But it was only in 1932 that these two advances were unified, and these new statistical recipes were provided with a systematic theoretical underpinning. In a stroke of mathematical genius, John von Neumann axiomatized the theory of mathematical spaces equipped with linear structure and an inner product, a type of space that was finding extensive use by David Hilbert's school in Göttingen. Von Neumann labeled any such space that is topologically complete (i.e., containing limit points for all Cauchy sequences) a "Hilbert space." He then went on to show how vectors in a Hilbert space can represent the states of quantum systems and how linear operators on a Hilbert space can represent the quantities, or "observables," of the system. With von Neumann's formalism in hand, quantum theorists had a precise mathematical justification for their statistical recipes. Quantum theory had entered the domain of *mathematical physics*.

However, von Neumann's formalization of quantum theory has yielded a false sense of conceptual clarity, for von Neumann's formalization pushes back but does not solve the basic interpretive problems of quantum theory. In particular, his formalism provides accurate statistical predictions, but only if it is severely limited in its application. Indeed, we still do not know how to apply quantum mechanics to individual systems, to macroscopic systems, or, a fortiori, to "observers" like ourselves.

Furthermore, although the Hilbert space formalism of quantum theory served as the framework for some of the twentieth century's greatest scientific achievements (e.g., the standard model of particle physics), it is not clear that it will prove serviceable in the attempt to unify quantum theory and the general theory of relativity. In fact, according to some notable physicists—such as Penrose [4] and Isham (see Chapter 3 in this volume)—the Hilbert space formalism might itself be implicated in our seeming inability to find a conceptual unification of our best two physical theories.

With these facts in mind, the authors of this book engage critically with the very mathematical foundations of quantum theory. In fact, not a single contributor to this book accepts, uncritically, the "standard formalism"—the Hilbert space formalism—as a background framework with which to pursue conceptual and empirical questions. Rather, a consistent theme of this volume is that we need to think creatively, not just *within the current framework*, but *beyond* it; that is, we need to think creatively about how to transcend, or at least reenvision, the current framework.

As mentioned, the authors of this volume approach this task from a broad range of perspectives. Several of them (e.g., Baez and Lauda; Coecke; Döring; Heunen, Landsman, and Spitters; Isham) attack the problem using the tools of category theory, the theory of mathematical structures attributable primarily to Samuel Eilenberg and Saunders Mac Lane (see, for example, [5]). Others (e.g., Redéi, Summers) make extensive use of the theory of operator algebras, a theory originally developed by von Neumann himself that has found application in formalizing quantum field theory and (deformation) quantization theory. Yet others (e.g., Dakić and Brukner, Hardy) prefer to reduce mathematical assumptions to a bare minimum in the interest of displaying more vividly the physical content of quantum theory and more general probabilistic theories. Thus, although the underlying motivations are analogous, the tools employed are quite diverse.

I.2 Categorical Approaches to Quantum Theory

In recent years, category theory has found many uses in physics and, indeed, in many of the exact sciences. This volume contains a representative sample of cutting-edge uses of category theory in the *foundations of physics*.

In this book, three sorts of category-theoretic approaches to the foundations of physics are represented: an n-categorical approach (Baez and Lauda), a monoidal categorical approach (Coecke), and a topos theoretical approach (Isham; Döring; Heunen, Landsman, and Spitters). Anyone who is acquainted with category theory will recognize immediately that these approaches need not be seen as opposed or even as disjoint. Indeed, they are in many ways mutually reinforcing and might even someday be unified (e.g., by some notion of a weak monoidal n-topos).

I.2.1 n-Categorical Physics

In their magisterial "A Prehistory of n-Categorical Physics," Chapter 1 in this volume, John C. Baez and Aaron D. Lauda recount in this volume the ways in which n-category theory has entered into physics and discuss many of the ways in which n-categories might play a role in the physical theories of the future. But why, you might ask, should we think that n-categories are a good place to look for some new insight into the very basic structures of the physical world? As Baez and Lauda point out, the theory of n-categories is itself based on a perspective-changing idea: the idea that what might be seen as an object from one point of view might be seen as a process from another point of view. For the simplest example of this "Copernican revolution" of mathematical framework, consider the example of a group, that is, a set G equipped with a binary product and an identity element $e \in G$ satisfying certain equations. Because we frequently think of categories on the model of concrete categories (i.e., categories of sets equipped with structure), it comes as a bit of a surprise to realize that a group is *itself* an example of a category. In particular, a group G is a category with one object (call it whatever you wish, say $*$) and whose arrows are elements of G.

Such a change of perspective might seem rather minor, but we should not minimize the amount of insight that can be gained by seeing a familiar object in a new guise. For example, once we see a group as a category, we can also see a group representation as a certain sort of functor, that is to say, a functor into the category HILB of Hilbert spaces. But now these group representations themselves naturally form a category, and we can consider the arrows in this category, what are usually called "intertwiners." With this new perspective on groups, Baez and Lauda point out that Feynman diagrams and Penrose spin networks are both examples of categories of group representations with intertwiners as arrows.

Baez and Lauda go on to discuss some of the most interesting recent developments in which category theory, and n-category theory in particular, promises to open new vistas. Among these developments, they discuss topological quantum field theories and quantum groups. They also briefly discuss Baez's own "periodic table" of n-categories, which neatly characterizes the zoology of higher categories.

I.2.2 Quantum Theory in Monoidal Categories

As briefly mentioned, the category HILB of Hilbert spaces plays a central role in quantum physics. We now expect, however, that quantum theory will play a central role in the computation theory of the future. After all, physical computers are made of objects that obey the laws of quantum mechanics.

It is well known that a quantum computer behaves differently than a classical computer; it is the differences in behavior that account, for example, for the fact that a quantum computer should be able to solve some problems more efficiently than any classical computer. But theoretical computer science is wont to abstract away from the nitty-gritty details of physical systems. In most cases, the computer scientist needs only to know the *structural* properties of the systems at his disposal; it is these structural properties that determine how such systems might be used to implement computations or other information-theoretic protocols.

It is no surprise, then, that theoretical computer scientists have led the way in describing the structural features of quantum systems. It is also no surprise that theoretical computer scientists have found it useful to use notions from category theory in describing these structures.

In "A Universe of Processes and Some of Its Guises," Chapter 2 in this volume, Bob Coecke provides a blueprint of a universe governed by quantum mechanics. Intriguingly, however, we see this universe through the eyes of a computer scientist: we do not see waves, particles, or any other concrete manifestation of physical processes. Rather, by means of a diagrammatic calculus, Coecke displays the very structures of the processes that are permitted (and forbidden) by the laws of quantum theory.

What is perhaps most striking about Coecke's approach is the sheer ratio of results to assumptions. From an extremely Spartan set of assumptions about how processes can combine (both vertically and horizontally), Coecke is able to reproduce all of the central results of quantum information science (in a broadly construed sense that includes "von Neumann measurement").

Another noteworthy aspect of Coecke's chapter is his discussion of the relation of categorical quantum mechanics (in its monoidal category guise) to other traditional approaches to the mathematical foundations of quantum mechanics (e.g., quantum logic, convex sets, C^*-algebras). Here, we get a "compare and contrast" from a researcher who has worked on both sides of the fence, first as a member of the Brussels school (directly descended from the Geneva school of Jauch and Piron) and more recently as a cofounder (with Samson Abramsky) and leader of the categorical approach to quantum computation. Thus, this chapter is absolutely mandatory reading for anyone interested in the fate of our attempts to *understand* the formalism of quantum theory and its utility in describing the processes that occur in our world.

I.2.3 Quantum Theory in Toposes

What is so radical about quantum theory? Perhaps the first thing to spring to mind is *indeterminism*: quantum theory describes a world in which the future is not determined by the past. With a bit more sophistication, one might claim that the most radical feature of quantum theory is *nonlocality*: quantum theory describes a world in which subtle dependency relations exist between events that occur in distant regions of space.

Another suggestion, originally put forward by Birkhoff and von Neumann [2], and later taken up by the philosopher Hilary Putnam [8], is that quantum theory overturns the laws of classical logic. According to this proposal, the rules of classical (formal) logic—in particular, the distribution postulate (of conjunction over disjunction)—lead to conclusions in conflict with the predictions of quantum theory. Thus, the new physics requires a revolution in logic. Indeed, Putnam went on to claim that quantum theory's relation to logic is directly analogous to general relativity's relation to geometry: just as general relativity forces us to abandon Euclidean geometry, so quantum theory forces us to abandon classical logic.

But this proposal has not found many advocates—even Birkhoff, von Neumann, and Putnam eventually abandoned the idea. Neither has quantum logic catalyzed progress within physics or suggested routes toward the unification of quantum theory and general relativity. Even if quantum logic has not been shown to be wrong, it has proved to be *mathematically sterile*: it fails to link up in interesting ways with mainstream developments in mathematical physics.

The central motivating idea behind quantum logic is that the quantum revolution is a thoroughgoing *conceptual* revolution; that is, that it requires us to revise some of the *constitutive* concepts of our worldview. The idea itself is intriguing and perhaps even plausible. Thus, we turn with great interest to a recent proposal by Jeremy Butterfield and Chris Isham [3]. According to the Butterfield-Isham proposal, quantum mechanics requires us to replace not only classical logic but also the entire classical mathematical universe—as articulated in twentieth-century mathematical logic and set theory—with a more general universe of sets, namely a *topos*. It is true that such a replacement would also necessitate a replacement of classical logic but not, á la von Neumann, with a nondistributive logic. Rather, the internal logic of a topos is intuitionistic logic, where the law of excluded middle fails.

Three of the chapters in this book (Chapters 3, 4, and 6)—by Isham; Döring; and Heunen, Landsman, and Spitters—push the Butterfield-Isham idea even further. As we will see, the underlying idea of these approaches is strikingly similar to Putnam's, although it is executed within an infinitely richer and more fruitful mathematical context.

The chapters in this book represent two distinct approaches to using topos theory in the foundations of physics: the approach of Döring and Isham and the approach of Heunen, Landsman, and Spitters. (Both approaches have been developed extensively in the literature, and I refer the reader to the references within the chapters in this book.) Although there are several divergences in implementation between the Döring-Isham approach and the Heunen-Landsman-Spitters approach, the underlying idea is similar and in both cases would amount to nothing less than a Copernican revolution.

The idea of adopting a new mathematical universe is so radical and profound that one cannot appreciate it without immersing oneself in these works. (Of course, it would also help to spend some time learning background rudiments of topos theory; for this I recommend the book by Mac Lane and Moerdijk [6].) Rather than attempt to summarize the content of these chapters, I recommend that the reader begin by reading Isham's chapter, which provides a lucid motivation and discussion of the framework. The reader may then wish to proceed to the more technically demanding chapters by Heunen et al. and by Döring. Finally, in reading Döring's chapter, the reader can gain further insight by referring to de Groote's chapter,[1] Chapter 5, which carefully articulates some of the background mathematics needed to generalize familiar notions from the classical universe of sets to the quantum topos.

I.3 Operator Algebras

Since the 1960s, it has been appreciated that the theory of operator algebras (especially C^*- and von Neumann algebras) provides a natural generalization of the Hilbert space formalism and is especially suitable for formalizing quantum field theories, or quantum theories with superselection rules. More recently, operator algebras have been applied to the task of clarifying conceptual issues. In this vein, see especially the work on nonlocality carried out by Summers [11] and the work on quantum logic carried out by Rédei [9]. Summers and Redéi continue this sort of foundational work in their chapters in this book (Chapters 7 and 8, respectively). Summers addresses the vacuum state in relativistic quantum field theory (QFT) in his chapter, "Yet More Ado about Nothing: The Remarkable Relativistic Vacuum State," whereas Redéi examines Einstein's notion of "separability" of physical systems in his chapter, "Einstein Meets von Neumann: Locality and Operational Independence in Algebraic Quantum Field Theory."

In his chapter, Summers aims to characterize properties of the vacuum in relativistic QFT in a mathematically precise way. He begins with the standard

[1] Published posthumously. See note in Chapter 5.

characterization, which involves both symmetries (the vacuum as invariant state) and energy conditions (the vacuum as lowest energy state). He then points out that these characterizations do not straightforwardly generalize to QFT on curved spacetimes. Thus, we stand in need of a more mathematically nuanced characterization of the vacuum.

According to Summers, the primary tool needed for this characterization is the Tomita-Takesaki modular theory, in particular, the geometrical interpretation of modular theory provided by Bisognano and Wichmann. However, Summers proceeds to recount a more ambitious program that he and his collaborators have undertaken, a program that would use modular symmetries as a basis from which the very structure of spacetime could be recovered. As Summers points out, such a reconstruction would have profound conceptual implications. Indeed, one is tempted to say that the success of such a program would be a partial vindication of Leibniz-Machian relationalism about spacetime. But whether or not the reconstruction supports certain philosophical views about the nature of spacetime, a clearer understanding of the vacuum is crucial for the development of future physics, especially because future physical theories will most certainly not posit a fixed-background Minkowski spacetime structure.

Summers also discusses the fact—without mentioning explicitly that it was first proved by himself and Reinhard Werner—that the vacuum state is nonlocal and indeed violates Bell's inequality maximally relative to measurements that can be performed in tangent spacetime wedges. In doing so, Summers notes the importance of making fine-grained distinctions between different types of nonlocality. This theme is treated at length in the chapter by Rédei.

Rédei begins in a historical vein by discussing Einstein's worries about quantum theory, in particular his notion of "separability" of physical systems. Although Einstein's objections to indeterminism are better known (witness: "God does not roll dice"), Einstein seems to have lost even more sleep over the issue of nonlocality. Indeed, it seems he thought that quantum nonlocality would make physics impossible!

Rédei distills from Einstein's writings a set of criteria that any theory must satisfy to be consistent with the principle of locality. He then proceeds to argue that relativistic QFT does in fact satisfy these criteria! Moreover, Rédei's arguments are far from speculative—or, as some might dismissively say, "philosophical." Rather, Rédei proceeds in a highly mathematical spirit: he translates the criteria into precise mathematical claims, and then he employs the tools of operator algebras in an attempt to demonstrate that the criteria are satisfied. The net result is a paradigm example of mathematical innovation in the service of conceptual clarification.

I.4 Behind the Hilbert Space Formalism

We have seen that several of the chapters in this book take well-developed (or independently developing) mathematical theories and apply them in innovative ways to the foundations of physics. Such an approach is characteristic of mathematical physics. This book, however, also represents a second approach, an approach more characteristic of theoretical physics. In particular, theoretical physicists begin from explicitly physical principles, rather than from mathematical assumptions, and then attempt to formulate these physical principles in as transparent a fashion as possible, using mathematical formalism when it might help achieve that goal. The three chapters by Dakić and Brukner (Chapter 9), by Bub (Chapter 10), and by Hardy (Chapter 11) exemplify this second methodology.

In their chapter, "Quantum Theory and Beyond: Is Entanglement Special?," Boriroje Dakić and Časlav Brukner aim to clarify the fundamental physical principles underlying quantum theory; in doing so, they keep in firm view the relationship between quantum theory and potential future theories in physics. The authors begin by recounting several recent attempts to derive the formalism of quantum theory from physical principles that were motivated by Einstein's derivation of special relativity. As they note, such derivations ought to be subjected to severe critical scrutiny because thinking that quantum theory "must be true" could easily impede the development of successor theories and could easily blind us to ways in which quantum theory could be modified or superseded.

Nonetheless, Dakić and Brukner prove that quantum theory is the *unique* theory that describes entangled states and that satisfies their other physical principles. This striking result displays a sort of robustness of the central features of quantum theory: to the extent that the basic physical principles are justified, we can expect *any* future theory to incorporate, rather than supersede, quantum theory.

This same sensitivity to quantum mechanics as a potentially replaceable theory is displayed throughout the chapter by Lucien Hardy. In "Foliable Operational Structures for General Probabilistic Theories," Hardy in essence provides a parameterization of theories in terms of a crucial equation involving two variables, K and N. In this parameterization, classical mechanics is characterized by the equation $K = N$, whereas quantum mechanics is characterized by the equation $K = N^2$. This leaves open the possibility of alternative theories, or even possible successor theories, of greater conceptual intricacy. Our past and current theories are only at the very low end of an infinite hierarchy of increasingly complex theories.

Hardy's chapter also pays special attention to the generalizability, or projectability into the future, of our theories. In particular, Hardy constructs his generalized probabilistic framework without reliance on a notion of fixed background time. As a result, the framework stands ready for application to *relativistic* contexts. But more is true: Hardy develops his framework with an eye on synthesis of general relativity and quantum mechanics, a context in which causal structure is flexible enough that it might be adapted to contexts where even it is subject to quantum indeterminacy.

In his chapter, "Is von Neumann's 'No Hidden Variables' Proof Silly?", Jeffrey Bub, takes up the question of whether the Hilbert space formalism of quantum mechanics is complete. That is, do all states correspond to vectors (or density operators), or could there be "hidden variables"? This question was supposedly answered in the negative in 1932 by von Neumann's no hidden variables proof. If this argument were valid, there would be a strong sense in which the interpretive problems of quantum mechanics could *not* be solved by means of technical innovation—for example, by providing a more complete formalism.

But von Neumann's argument has not convinced everyone. In particular, John Bell [1] and, subsequently, David Mermin [7], argued that von Neumann's result is based on an illicit assumption—in particular, that von Neumann imposes unrealistic constraints on the mathematical representation of hidden variables. These critiques of von

Neumann's result were motivated by and, in turn, provide support for hidden variable programs, such as Bohmian mechanics.

Bub argues, however, that Bell and Mermin's criticism is off the mark. Rather, claims Bub, von Neumann states quite clearly that an operator $A + B$ has no direct physical significance in cases where A and B are incompatible (i.e., not simultaneously measurable). Read from this perspective, von Neumann intends to show not that hidden variables are impossible *tout court* but rather that hidden variables are inconsistent with the way that quantum mechanics uses mathematical objects to represent physical objects. But then the possibility opens that intuitive desiderata for a physical theory of micro-objects (e.g., determinism) could be satisfied only by overhauling the Hilbert space formalism.

Bub closes his chapter on this suggestive note, leaving it for the reader to judge whether it would be preferable to maintain the Hilbert space formalism, along with its puzzling interpretive consequences, or to attempt to replace it with some other formalism.

The book concludes with Chapter 12, an already famous article, "The Strong Free Will Theorem," by John H. Conway and Simon Kochen (reprinted in this volume with permission). But what has such an argument to do with the theme of the book—that is, with the theme of conceptual insight developing in tandem with mathematical insight? The careful reader will see that Conway and Kochen's argument proceeds independently of the standard formalism (i.e., Hilbert spaces) for quantum theory. That is, the authors do not take the Hilbert space formalism for granted and then draw out conceptual consequences regarding free will. Rather, they argue from simple, physically verifiable assumptions to the conclusion that if an experimenter has the freedom to choose what to measure, then particles have the freedom to choose what result to yield. The only input here from quantum mechanics is indirect: quantum mechanics predicts that Conway and Kochen's empirical assumptions are satisfied. Thus, if quantum mechanics is true, then Conway and Kochen's argument is sound.

We see, then, that Conway and Kochen's argument exemplifies the method of applying mathematical argument to the task of gaining new conceptual insight—insight, in this case, about the logical connection between certain statistical predictions (which are in fact made by quantum mechanics) and traditional metaphysical hypotheses (freedom of the will). If their argument is successful, then Conway and Kochen have provided us with insight that transcends the bounds of our current mathematical framework—hence, insight that will endure through the vicissitudes of scientific progress or revolutions.

In conclusion, the authors of this book were given carte blanche to employ as little or as much technical apparatus as they deemed necessary to advance conceptual understanding of the foundations of physics. For some of the authors, this meant employing highly sophisticated mathematical theories such as n-categories (Baez and Lauda), monoidal categories (Coecke), topos theory (Döring, Isham, Heunen et al.), or operator algebras (Rédei, Summers). For other authors, the emphasis lies more on examining the physical and conceptual motivation for the Hilbert space formalism (Dakić and Brukner) or on what might lie beyond the Hilbert space formalism (Bub, Hardy).

The liberty given to the authors means that for the reader, some of these chapters are technically demanding; even for those with previous technical training, these chapters should be approached with equal doses of patience and persistence. However, the technicalities seem to be demanded by the nature of the subject matter: quantum theory shows that conceptual insights and understanding do not come cheap, and the physical world does not come ready-made to be understood by the untrained human mind. Already it required the combined mathematical genius of Dirac and von Neumann, among others, to unify the various statistical recipes of the old quantum theory. The Hilbert space formalism has proved fruitful for many years and is partially responsible for some of the great advances of twentieth-century physics. But taking the Hilbert space formalism as a fixed, non-negotiable framework may also be partially responsible for our current predicament—both our troubles in interpreting quantum mechanics and the challenges of unifying quantum theory with the general theory of relativity. If this is the case, then it is imperative that we marshal the same sorts of resources that Dirac and von Neumann did; we must, indeed, employ our utmost mathematical creativity in our attempt to find an underlying intelligibility behind the physical phenomena.

It is with this aim in mind that the contributors present this collection to you, hoping to play some small role in the next quantum leap in our understanding of nature.

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