Game Theory

Covering both noncooperative and cooperative games, this comprehensive introduction to game theory also includes some advanced chapters on auctions, games with incomplete information, games with vector payoffs, stable matchings, and the bargaining set. Mathematically oriented, the book presents every theorem alongside a proof. The material is presented clearly and every concept is illustrated with concrete examples from a broad range of disciplines. With numerous exercises the book is a thorough and extensive guide to game theory from undergraduate through graduate courses in economics, mathematics, computer science, engineering, and life sciences to being an authoritative reference for researchers.

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To Michael Maschler
# Contents

- **Acknowledgments**  
- **Notations**  
- **Introduction**

## 1 The game of chess

1.1 Schematic description of the game  
1.2 Analysis and results  
1.3 Remarks  
1.4 Exercises

## 2 Utility theory

2.1 Preference relations and their representation  
2.2 Preference relations over uncertain outcomes: the model  
2.3 The axioms of utility theory  
2.4 The characterization theorem for utility functions  
2.5 Utility functions and affine transformations  
2.6 Infinite outcome set  
2.7 Attitude towards risk  
2.8 Subjective probability  
2.9 Discussion  
2.10 Remarks  
2.11 Exercises

## 3 Extensive-form games

3.1 An example  
3.2 Graphs and trees  
3.3 Game trees  
3.4 Chomp: David Gale’s game  
3.5 Games with chance moves  
3.6 Games with imperfect information  
3.7 Exercises
## Contents

<table>
<thead>
<tr>
<th>4</th>
<th>Strategic-form games</th>
<th>75</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>Examples and definition of strategic-form games</td>
<td>76</td>
</tr>
<tr>
<td>4.2</td>
<td>The relationship between the extensive form and the strategic form</td>
<td>82</td>
</tr>
<tr>
<td>4.3</td>
<td>Strategic-form games: solution concepts</td>
<td>84</td>
</tr>
<tr>
<td>4.4</td>
<td>Notation</td>
<td>85</td>
</tr>
<tr>
<td>4.5</td>
<td>Domination</td>
<td>85</td>
</tr>
<tr>
<td>4.6</td>
<td>Second-price auctions</td>
<td>91</td>
</tr>
<tr>
<td>4.7</td>
<td>The order of elimination of dominated strategies</td>
<td>95</td>
</tr>
<tr>
<td>4.8</td>
<td>Stability: Nash equilibrium</td>
<td>95</td>
</tr>
<tr>
<td>4.9</td>
<td>Properties of the Nash equilibrium</td>
<td>100</td>
</tr>
<tr>
<td>4.10</td>
<td>Security: the maxmin concept</td>
<td>102</td>
</tr>
<tr>
<td>4.11</td>
<td>The effect of elimination of dominated strategies</td>
<td>106</td>
</tr>
<tr>
<td>4.12</td>
<td>Two-player zero-sum games</td>
<td>110</td>
</tr>
<tr>
<td>4.13</td>
<td>Games with perfect information</td>
<td>118</td>
</tr>
<tr>
<td>4.14</td>
<td>Games on the unit square</td>
<td>121</td>
</tr>
<tr>
<td>4.15</td>
<td>Remarks</td>
<td>128</td>
</tr>
<tr>
<td>4.16</td>
<td>Exercises</td>
<td>128</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>5</th>
<th>Mixed strategies</th>
<th>144</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.1</td>
<td>The mixed extension of a strategic-form game</td>
<td>145</td>
</tr>
<tr>
<td>5.2</td>
<td>Computing equilibria in mixed strategies</td>
<td>152</td>
</tr>
<tr>
<td>5.3</td>
<td>The proof of Nash’s Theorem</td>
<td>166</td>
</tr>
<tr>
<td>5.4</td>
<td>Generalizing Nash’s Theorem</td>
<td>170</td>
</tr>
<tr>
<td>5.5</td>
<td>Utility theory and mixed strategies</td>
<td>172</td>
</tr>
<tr>
<td>5.6</td>
<td>The maxmin and the minmax in n-player games</td>
<td>176</td>
</tr>
<tr>
<td>5.7</td>
<td>Imperfect information: the value of information</td>
<td>180</td>
</tr>
<tr>
<td>5.8</td>
<td>Evolutionarily stable strategies</td>
<td>186</td>
</tr>
<tr>
<td>5.9</td>
<td>Remarks</td>
<td>194</td>
</tr>
<tr>
<td>5.10</td>
<td>Exercises</td>
<td>194</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>6</th>
<th>Behavior strategies and Kuhn’s Theorem</th>
<th>219</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.1</td>
<td>Behavior strategies</td>
<td>221</td>
</tr>
<tr>
<td>6.2</td>
<td>Kuhn’s Theorem</td>
<td>226</td>
</tr>
<tr>
<td>6.3</td>
<td>Equilibria in behavior strategies</td>
<td>235</td>
</tr>
<tr>
<td>6.4</td>
<td>Kuhn’s Theorem for infinite games</td>
<td>238</td>
</tr>
<tr>
<td>6.5</td>
<td>Remarks</td>
<td>243</td>
</tr>
<tr>
<td>6.6</td>
<td>Exercises</td>
<td>244</td>
</tr>
</tbody>
</table>
## Contents

### 7 Equilibrium refinements

7.1 Subgame perfect equilibrium 252
7.2 Rationality, backward induction, and forward induction 260
7.3 Perfect equilibrium 262
7.4 Sequential equilibrium 271
7.5 Remarks 284
7.6 Exercises 284

### 8 Correlated equilibria

8.1 Examples 301
8.2 Definition and properties of correlated equilibrium 305
8.3 Remarks 313
8.4 Exercises 313

### 9 Games with incomplete information and common priors

9.1 The Aumann model of incomplete information and the concept of knowledge 322
9.2 The Aumann model of incomplete information with beliefs 334
9.3 An infinite set of states of the world 344
9.4 The Harsanyi model of games with incomplete information 345
9.5 Incomplete information as a possible interpretation of mixed strategies 361
9.6 The common prior assumption: inconsistent beliefs 365
9.7 Remarks 367
9.8 Exercises 368

### 10 Games with incomplete information: the general model

10.1 Belief spaces 386
10.2 Belief and knowledge 391
10.3 Examples of belief spaces 394
10.4 Belief subspaces 400
10.5 Games with incomplete information 407
10.6 The concept of consistency 415
10.7 Remarks 423
10.8 Exercises 423

### 11 The universal belief space

11.1 Belief hierarchies 442
11.2 Types 450
## Contents

11.3 Definition of the universal belief space 453  
11.4 Remarks 456  
11.5 Exercises 456

12 Auctions 461  
12.1 Notation 464  
12.2 Common auction methods 464  
12.3 Definition of a sealed-bid auction with private values 465  
12.4 Equilibrium 468  
12.5 The symmetric model with independent private values 471  
12.6 The Envelope Theorem 484  
12.7 Risk aversion 488  
12.8 Mechanism design 492  
12.9 Individually rational mechanisms 500  
12.10 Finding the optimal mechanism 501  
12.11 Remarks 508  
12.12 Exercises 509

13 Repeated games 519  
13.1 The model 520  
13.2 Examples 521  
13.3 The $T$-stage repeated game 524  
13.4 Characterization of the set of equilibrium payoffs of the $T$-stage repeated game 530  
13.5 Infinitely repeated games 537  
13.6 The discounted game 542  
13.7 Uniform equilibrium 546  
13.8 Discussion 554  
13.9 Remarks 555  
13.10 Exercises 555

14 Repeated games with vector payoffs 569  
14.1 Notation 570  
14.2 The model 572  
14.3 Examples 573  
14.4 Connections between approachable and excludable sets 574  
14.5 A geometric condition for the approachability of a set 576  
14.6 Characterizations of convex approachable sets 585  
14.7 Application 1: Repeated games with incomplete information 590  
14.8 Application 2: Challenge the expert 600  
14.9 Discussion 606  
14.10 Remarks 607  
14.11 Exercises 608
### 15 Bargaining games

- **15.1 Notation**
- **15.2 The model**
- **15.3 Properties of the Nash solution**
- **15.4 Existence and uniqueness of the Nash solution**
- **15.5 Another characterization of the Nash solution**
- **15.6 The minimality of the properties of the Nash solution**
- **15.7 Critiques of the properties of the Nash solution**
- **15.8 Monotonicity properties**
- **15.9 Bargaining games with more than two players**
- **15.10 Remarks**
- **15.11 Exercises**

### 16 Coalitional games with transferable utility

- **16.1 Examples**
- **16.2 Strategic equivalence**
- **16.3 A game as a vector in a Euclidean space**
- **16.4 Special families of games**
- **16.5 Solution concepts**
- **16.6 Geometric representation of the set of imputations**
- **16.7 Remarks**
- **16.8 Exercises**

### 17 The core

- **17.1 Definition of the core**
- **17.2 Balanced collections of coalitions**
- **17.3 The Bondareva–Shapley Theorem**
- **17.4 Market games**
- **17.5 Additive games**
- **17.6 The consistency property of the core**
- **17.7 Convex games**
- **17.8 Spanning tree games**
- **17.9 Flow games**
- **17.10 The core for general coalitional structures**
- **17.11 Remarks**
- **17.12 Exercises**

### 18 The Shapley value

- **18.1 The Shapley properties**
- **18.2 Solutions satisfying some of the Shapley properties**
- **18.3 The definition and characterization of the Shapley value**
- **18.4 Examples**
## Contents

18.5 An alternative characterization of the Shapley value 760  
18.6 Application: the Shapley–Shubik power index 763  
18.7 Convex games 767  
18.8 The consistency of the Shapley value 768  
18.9 Remarks 774  
18.10 Exercises 774

19 The bargaining set 782

19.1 Definition of the bargaining set 784  
19.2 The bargaining set in two-player games 788  
19.3 The bargaining set in three-player games 788  
19.4 The bargaining set in convex games 794  
19.5 Discussion 797  
19.6 Remarks 798  
19.7 Exercises 798

20 The nucleolus 801

20.1 Definition of the nucleolus 802  
20.2 Nonemptiness and uniqueness of the nucleolus 805  
20.3 Properties of the nucleolus 809  
20.4 Computing the nucleolus 815  
20.5 Characterizing the prenucleolus 816  
20.6 The consistency of the nucleolus 823  
20.7 Weighted majority games 825  
20.8 The bankruptcy problem 831  
20.9 Discussion 842  
20.10 Remarks 843  
20.11 Exercises 844

21 Social choice 853

21.1 Social welfare functions 856  
21.2 Social choice functions 864  
21.3 Non-manipulability 871  
21.4 Discussion 873  
21.5 Remarks 874  
21.6 Exercises 874

22 Stable matching 884

22.1 The model 886  
22.2 Existence of stable matching: the men’s courtship algorithm 888  
22.3 The women’s courtship algorithm 890
## Contents

22.4 Comparing matchings 892  
22.5 Extensions 898  
22.6 Remarks 905  
22.7 Exercises 905

### 23 Appendices

<table>
<thead>
<tr>
<th>Appendix</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>23.1</td>
<td>Fixed point theorems</td>
<td>916</td>
</tr>
<tr>
<td>23.2</td>
<td>The Separating Hyperplane Theorem</td>
<td>943</td>
</tr>
<tr>
<td>23.3</td>
<td>Linear programming</td>
<td>945</td>
</tr>
<tr>
<td>23.4</td>
<td>Remarks</td>
<td>950</td>
</tr>
<tr>
<td>23.5</td>
<td>Exercises</td>
<td>950</td>
</tr>
</tbody>
</table>

References 958

Index 968
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The book makes use of a large number of notations; we have striven to stick to accepted notation and to be consistent throughout the book. The coordinates of a vector are always denoted by a subscript index, $x = (x_i)_{i=1}^n$, while the indices of the elements of sequences are always denoted by a superscript index, $x^1, x^2, \ldots$. The index of a player in a set of players is always denoted by a subscript index, while a time index (in repeated games) is always denoted by a superscript index. The end of the proof of a theorem is indicated by $\Box$, the end of an example is indicated by $\blacktriangle$, and the end of a remark is indicated by $\blacklozenge$.

For convenience we provide a list of the mathematical notation used throughout the book, accompanied by a short explanation and the pages on which they are formally defined. The notations that appear below are those that are used more than once.

<table>
<thead>
<tr>
<th>Notation</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>chance move in an extensive-form game</td>
<td>50</td>
</tr>
<tr>
<td>$\vec{0}$</td>
<td>origin of a Euclidean space</td>
<td>570</td>
</tr>
<tr>
<td>$\emptyset$</td>
<td>strategy used by a player who has no decision vertices in an extensive-form game</td>
<td>5</td>
</tr>
<tr>
<td>$\mathbf{1}_A$</td>
<td>function that is equal to 1 on event $A$ and to 0 otherwise</td>
<td>595</td>
</tr>
<tr>
<td>$2^Y$</td>
<td>collection of all subsets of $Y$</td>
<td>325</td>
</tr>
<tr>
<td>$</td>
<td>X</td>
<td>$</td>
</tr>
<tr>
<td>$|x|_{\infty}$</td>
<td>$L_\infty$ norm, $|x|<em>{\infty} := \max</em>{i=1,2,\ldots,n}</td>
<td>x_i</td>
</tr>
<tr>
<td>$|x|$</td>
<td>norm of a vector, $|x| := \sqrt{\sum_{i=1}^d (x_i)^2}$</td>
<td>570</td>
</tr>
<tr>
<td>$A \lor B$</td>
<td>maximum matching (for men) in a matching problem</td>
<td>895</td>
</tr>
<tr>
<td>$A \land B$</td>
<td>maximum matching (for women) in a matching problem</td>
<td>896</td>
</tr>
<tr>
<td>$A \subseteq B$</td>
<td>set $A$ contains set $B$ or is equal to it</td>
<td></td>
</tr>
<tr>
<td>$A \subset B$</td>
<td>set $A$ strictly contains set $B$</td>
<td></td>
</tr>
<tr>
<td>$\langle x, y \rangle$</td>
<td>inner product</td>
<td>570</td>
</tr>
<tr>
<td>$\langle x_0, \ldots, x_k \rangle$</td>
<td>$k$-dimensional simplex</td>
<td>920</td>
</tr>
<tr>
<td>$\succeq_i$</td>
<td>preference relation of player $i$</td>
<td>14</td>
</tr>
<tr>
<td>$\succ_i$</td>
<td>strict preference relation of player $i$</td>
<td>10</td>
</tr>
<tr>
<td>$\approx_i$</td>
<td>indifference relation of player $i$</td>
<td>10, 897</td>
</tr>
<tr>
<td>$\succeq_p$</td>
<td>preference relation of an individual</td>
<td>857</td>
</tr>
<tr>
<td>$\succ Q$</td>
<td>strict preference relation of society</td>
<td>857</td>
</tr>
<tr>
<td>$\approx Q$</td>
<td>indifference relation of society</td>
<td>857</td>
</tr>
<tr>
<td>$x \geq y$</td>
<td>$x_k \geq y_k$ for each coordinate $k$, where $x, y$ are vectors in a Euclidean space</td>
<td>625</td>
</tr>
<tr>
<td>$x &gt; y$</td>
<td>$x \geq y$ and $x \neq y$</td>
<td>625</td>
</tr>
</tbody>
</table>
Notations

\( x \gg y \quad x_k > y_k \) for each coordinate \( k \), where \( x, y \) are vectors in a Euclidean space 625

\( x + y \) sum of vectors in a Euclidean space, \((x + y)_k := x_k + y_k \) 625

\( xy \) coordinatewise product of vectors in a Euclidean space, \((xy)_k := x_k y_k \) 625

\( x + S \) \( x + S := \{x + s : s \in S\} \), where \( x \in \mathbb{R}^d \) and \( S \subseteq \mathbb{R}^d \) 625

\( xS \) \( xS := \{xs : s \in S\} \), where \( x \in \mathbb{R}^d \) and \( S \subseteq \mathbb{R}^d \) 625

\( c \) smallest integer greater than or equal to \( c \) 534

\( \lfloor c \rfloor \) largest integer less than or equal to \( c \) 534

\( x^\top \) transpose of a vector, column vector that corresponds to row vector \( x \) 571

\( \arg\max_{x \in X} f(x) \) set of all \( x \) where function \( f \) attains its maximum in the set \( X \) 125, 625

\( a(i) \) producer \( i \)'s initial endowment in a market 703

\( A \) set of actions in a decision problem with experts 601

\( A \) set of alternatives 856

\( A_i \) player \( i \)'s action set in an extensive-form game, \( A_i := \bigcup_{j=1}^k A(U^i_j) \) 221

\( A_k \) possible outcome of a game 13

\( A(x) \) set of available actions at vertex \( x \) in an extensive-form game 44

\( A(U_i) \) set of available actions at information set \( U_i \) of player \( i \) in an extensive-form game 54

\( b_i \) buyer \( i \)'s bid in an auction 91, 466

\( b(S) \) \( b(S) = \sum_{i \in S} b_i \) where \( b \in \mathbb{R}^N \) 669

\( \text{br}_1(y) \) Player I's set of best replies to strategy \( y \) 125

\( \text{br}_2(x) \) Player II's set of best replies to strategy \( x \) 125

\( B_i \) player \( i \)'s belief operator 392

\( B^p_i \) set of states of the world in which the probability that player \( i \) ascribes to event \( E \) is at least \( p \), \( B^p_i(E) := \{\omega \in Y : \pi_i(E \mid \omega) \geq p\} \) 426

\( \text{BZ}_i(N; v) \) Banzhaf value of a coalitional game 780

\( \mathcal{B} \) coalitional structure 673

\( B^T_i \) set of behavior strategies of player \( i \) in a \( T \)-repeated game 525

\( B^\infty_i \) set of behavior strategies of player \( i \) in an infinitely repeated game 538

\( c \) coalitional function of a cost game 661

\( c_+ \) maximum of \( c \) and 0 840

\( c_i \) \( c_i(v_i) := v_i - \frac{1}{f(v_i)} \) 501

\( C \) function that dictates the amount that each buyer pays given the vector of bids in an auction 466
### Notations

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C(x)$</td>
<td>set of children of vertex $x$ in an extensive-form game</td>
<td>5</td>
</tr>
<tr>
<td>$C(N, v)$</td>
<td>core of a coalitional game</td>
<td>687</td>
</tr>
<tr>
<td>$C(N, v; B)$</td>
<td>core for a coalitional structure</td>
<td>732</td>
</tr>
<tr>
<td>$\text{conv}{x_1, \ldots, x_K}$</td>
<td>smallest convex set that contains the vectors ${x_1, \ldots, x_K}$</td>
<td>530, 625, 917</td>
</tr>
<tr>
<td>$d$</td>
<td>disagreement point of a bargaining game</td>
<td>625</td>
</tr>
<tr>
<td>$d_i$</td>
<td>debt to creditor $i$ in a bankruptcy problem</td>
<td>833</td>
</tr>
<tr>
<td>$d^t$</td>
<td>distance between average payoff and target set</td>
<td>581</td>
</tr>
<tr>
<td>$d(x, y)$</td>
<td>Euclidean distance between two vectors in Euclidean space</td>
<td>571</td>
</tr>
<tr>
<td>$d(x, S)$</td>
<td>Euclidean distance between point and set</td>
<td>571</td>
</tr>
<tr>
<td>$D(\alpha, x)$</td>
<td>collection of coalitions whose excess is at least $\alpha$, $D(\alpha, x) := {S \subseteq N, S \neq \emptyset: e(S, x) \geq \alpha}$</td>
<td>818</td>
</tr>
<tr>
<td>$e(S, x)$</td>
<td>excess of coalition $S$, $e(S, x) := v(S) - x(S)$</td>
<td>802</td>
</tr>
<tr>
<td>$E$</td>
<td>set of vertices of a graph</td>
<td>41, 43</td>
</tr>
<tr>
<td>$E$</td>
<td>estate of bankrupt entity in a bankruptcy problem</td>
<td>833</td>
</tr>
<tr>
<td>$E$</td>
<td>set of experts in a decision problem with experts</td>
<td>601</td>
</tr>
<tr>
<td>$F$</td>
<td>set of feasible payoffs in a repeated game</td>
<td>530, 578</td>
</tr>
<tr>
<td>$F$</td>
<td>social welfare function</td>
<td>857</td>
</tr>
<tr>
<td>$F_i$</td>
<td>cumulative distribution function of buyer $i$’s private values in an auction</td>
<td>466</td>
</tr>
<tr>
<td>$F_i(\omega)$</td>
<td>atom of the partition $\mathcal{F}_i$ that contains $\omega$</td>
<td>324</td>
</tr>
<tr>
<td>$F^N$</td>
<td>cumulative distribution function of joint distribution of vector of private values in an auction</td>
<td>466</td>
</tr>
<tr>
<td>$\mathcal{F}$</td>
<td>collection of all subgames in the game of chess</td>
<td>5</td>
</tr>
<tr>
<td>$\mathcal{F}$</td>
<td>family of bargaining games</td>
<td>625</td>
</tr>
<tr>
<td>$\mathcal{F}^N$</td>
<td>family of bargaining games with set of players $N$</td>
<td>650</td>
</tr>
<tr>
<td>$\mathcal{F}_d$</td>
<td>family of bargaining games in $\mathcal{F}$ where the set of alternatives is comprehensive and all alternatives are at least as good as the disagreement point, which is $(0, 0)$</td>
<td>644</td>
</tr>
<tr>
<td>$\mathcal{F}_i$</td>
<td>player $i$’s information in an Aumann model of incomplete information</td>
<td>323</td>
</tr>
<tr>
<td>$g^T$</td>
<td>average payoff up to stage $T$ (including) in a repeated game</td>
<td>572</td>
</tr>
<tr>
<td>$G$</td>
<td>graph</td>
<td>41</td>
</tr>
<tr>
<td>$G$</td>
<td>social choice function</td>
<td>865</td>
</tr>
<tr>
<td>$h$</td>
<td>history of a repeated game</td>
<td>525</td>
</tr>
<tr>
<td>$h_i$</td>
<td>history at stage $i$ of a repeated game</td>
<td>602</td>
</tr>
<tr>
<td>$H(t)$</td>
<td>set of $t$-stage histories of a repeated game</td>
<td>525, 601</td>
</tr>
<tr>
<td>$H(\infty)$</td>
<td>set of plays in an infinitely repeated game</td>
<td>538</td>
</tr>
<tr>
<td>$H(\alpha, \beta)$</td>
<td>hyperplane, $H(\alpha, \beta) := {x \in \mathbb{R}^d: \langle \alpha, x \rangle = \beta}$</td>
<td>577, 943</td>
</tr>
<tr>
<td>$H^+(\alpha, \beta)$</td>
<td>half-space, $H^+(\alpha, \beta) := {x \in \mathbb{R}^d: \langle \alpha, x \rangle \geq \beta}$</td>
<td>577, 943</td>
</tr>
<tr>
<td>$H^-(\alpha, \beta)$</td>
<td>half-space, $H^-(\alpha, \beta) := {x \in \mathbb{R}^d: \langle \alpha, x \rangle \leq \beta}$</td>
<td>577, 943</td>
</tr>
<tr>
<td>$i$</td>
<td>player</td>
<td></td>
</tr>
<tr>
<td>$-i$</td>
<td>set of all players except of player $i$</td>
<td></td>
</tr>
</tbody>
</table>

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### Notations

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$I$</td>
<td>function that dictates the winner of an auction given the vector of bids</td>
</tr>
<tr>
<td>$J$</td>
<td>number of lotteries that compose a compound lottery</td>
</tr>
<tr>
<td>$J(x)$</td>
<td>player who chooses a move at vertex $x$ of an extensive-form game</td>
</tr>
<tr>
<td>$-k$</td>
<td>player who is not $k$ in a two-player game</td>
</tr>
<tr>
<td>$k_i$</td>
<td>number of information sets of player $i$ in an extensive-form game</td>
</tr>
<tr>
<td>$K$</td>
<td>number of outcomes of a game</td>
</tr>
<tr>
<td>$K_i$</td>
<td>player $i$’s knowledge operator</td>
</tr>
<tr>
<td>$KS$</td>
<td>Kalai–Smorodinsky solution to bargaining games</td>
</tr>
<tr>
<td>$L$</td>
<td>lottery: $L = [p_1(A_1), p_2(A_2), \ldots, p_K(A_K)]$</td>
</tr>
<tr>
<td>$L$</td>
<td>number of commodities in a market</td>
</tr>
<tr>
<td>$\hat{L}$</td>
<td>compound lottery: $\hat{L} = [q_1(L_1), \ldots, q_J(L_J)]$</td>
</tr>
<tr>
<td>$\mathcal{L}$</td>
<td>set of lotteries</td>
</tr>
<tr>
<td>$\hat{\mathcal{L}}$</td>
<td>set of compound lotteries</td>
</tr>
<tr>
<td>$m(\epsilon)$</td>
<td>minimal coordinate of vector $\epsilon$</td>
</tr>
<tr>
<td>$m_i$</td>
<td>number of pure strategies of player $i$</td>
</tr>
<tr>
<td>$m_i(S)$</td>
<td>highest possible payoff to player $i$ in a bargaining game</td>
</tr>
<tr>
<td>$M$</td>
<td>maximal absolute value of a payoff in a game</td>
</tr>
<tr>
<td>$M_{m,l}$</td>
<td>space of matrices of dimension $m \times l$</td>
</tr>
<tr>
<td>$M(\epsilon)$</td>
<td>maximal coordinate of vector $\epsilon$</td>
</tr>
<tr>
<td>$\mathcal{M}(N; v; B)$</td>
<td>bargaining set for coalitional structure $B$</td>
</tr>
<tr>
<td>$n$</td>
<td>number of players</td>
</tr>
<tr>
<td>$n$</td>
<td>number of buyers in an auction</td>
</tr>
<tr>
<td>$n_x$</td>
<td>number of vertices in subgame $\Gamma(x)$</td>
</tr>
<tr>
<td>$N$</td>
<td>set of players</td>
</tr>
<tr>
<td>$N$</td>
<td>set of buyers in an auction</td>
</tr>
<tr>
<td>$N$</td>
<td>set of individuals</td>
</tr>
<tr>
<td>$N$</td>
<td>set of producers in a market</td>
</tr>
<tr>
<td>$\mathbb{N}$</td>
<td>set of natural numbers, $\mathbb{N} := {1, 2, 3, \ldots}$</td>
</tr>
<tr>
<td>$\mathcal{N}$</td>
<td>$\mathcal{N}(S, d)$, Nash’s solution to bargaining games</td>
</tr>
<tr>
<td>$\mathcal{N}(N; v)$</td>
<td>nucleolus of a coalitional game</td>
</tr>
<tr>
<td>$\mathcal{N}(N; v; B)$</td>
<td>nucleolus of a coalitional game for coalitional structure $B$</td>
</tr>
<tr>
<td>$\mathcal{N}(N; v; K)$</td>
<td>nucleolus relative to set $K$</td>
</tr>
<tr>
<td>$O$</td>
<td>set of outcomes</td>
</tr>
<tr>
<td>$p$</td>
<td>common prior in a Harsanyi game with incomplete information</td>
</tr>
<tr>
<td>$p_k$</td>
<td>probability that the outcome of lottery $L$ is $A_k$</td>
</tr>
<tr>
<td>$p_x$</td>
<td>probability distribution over actions at chance move $x$</td>
</tr>
<tr>
<td>$P$</td>
<td>binary relation</td>
</tr>
</tbody>
</table>
### Notations

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$P$</td>
<td>set of all weakly balancing weights for collection $D^*$ of all coalitions</td>
</tr>
<tr>
<td>$P$</td>
<td>common prior in an Aumann model of incomplete information</td>
</tr>
<tr>
<td>$P_\sigma(x)$</td>
<td>probability that the play reaches vertex $x$ when the players implement strategy vector $\sigma$ in an extensive-form game</td>
</tr>
<tr>
<td>$P_\sigma(U)$</td>
<td>probability that the play reaches a vertex in information set $U$ when the players implement strategy vector $\sigma$ in an extensive-form game</td>
</tr>
<tr>
<td>$P^N$</td>
<td>vector of preference relations</td>
</tr>
<tr>
<td>$PO(S)$</td>
<td>set of efficient (Pareto optimal) points in $S$</td>
</tr>
<tr>
<td>$PO^W(S)$</td>
<td>set of weakly efficient points in $S$</td>
</tr>
<tr>
<td>$\mathcal{P}(A)$</td>
<td>set of all strict preference relations over a set of alternatives $A$</td>
</tr>
<tr>
<td>$\mathcal{P}(N)$</td>
<td>collection of nonempty subsets of $N$, $\mathcal{P}(N) := {S \subseteq N, S \neq \emptyset}$</td>
</tr>
<tr>
<td>$\mathcal{P}^*(A)$</td>
<td>set of all preference relations over a set of alternatives $A$</td>
</tr>
<tr>
<td>$\mathcal{P}_N(N; v)$</td>
<td>prenucleolus of a coalitional game</td>
</tr>
<tr>
<td>$\mathcal{P}_N(N; v; \mathcal{B})$</td>
<td>prenucleolus of a coalitional game for coalitional structure $\mathcal{B}$</td>
</tr>
<tr>
<td>$q$</td>
<td>quota in a weighted majority game</td>
</tr>
<tr>
<td>$q(w)$</td>
<td>minimal weight of a winning coalition in a weighted majority game, $q(w) := \min_{S \in W} w(S)$</td>
</tr>
<tr>
<td>$\mathbb{Q}_{++}$</td>
<td>set of positive rational numbers</td>
</tr>
<tr>
<td>$r_k$</td>
<td>total probability that the result of a compound lottery is $A_k$</td>
</tr>
<tr>
<td>$R_1(p)$</td>
<td>set of possible payoffs when Player 1 plays mixed action $p$, $R_1(p) := {puq^r : q \in \Delta(J)}$</td>
</tr>
<tr>
<td>$R_2(p)$</td>
<td>set of possible payoffs when Player 2 plays mixed action $q$, $R_2(p) := {puq^r : q \in \Delta(I)}$</td>
</tr>
<tr>
<td>$\mathbb{R}$</td>
<td>real line</td>
</tr>
<tr>
<td>$\mathbb{R}_+$</td>
<td>set of nonnegative numbers</td>
</tr>
<tr>
<td>$\mathbb{R}_{++}$</td>
<td>set of positive numbers</td>
</tr>
<tr>
<td>$\mathbb{R}^n$</td>
<td>$n$-dimensional Euclidean space</td>
</tr>
<tr>
<td>$\mathbb{R}_{++}^n$</td>
<td>nonnegative orthant in an $n$-dimensional Euclidean space, $\mathbb{R}_{++}^n := {x \in \mathbb{R}^n : x_i \geq 0, \forall i = 1, 2, \ldots, n}$</td>
</tr>
<tr>
<td>$\mathbb{R}^S$</td>
<td>$</td>
</tr>
<tr>
<td>range($G$)</td>
<td>range of a social choice function</td>
</tr>
<tr>
<td>$s$</td>
<td>strategy vector</td>
</tr>
<tr>
<td>$s$</td>
<td>function that assigns a state of nature to each state of the world</td>
</tr>
<tr>
<td>$s'$</td>
<td>action vector played at stage $t$ of a repeated game</td>
</tr>
<tr>
<td>$s_i$</td>
<td>strategy of player $i$</td>
</tr>
</tbody>
</table>
### Notations

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_t$</td>
<td>state of nature that corresponds to type vector $t$ in a Harsanyi game with incomplete information</td>
<td>347</td>
</tr>
<tr>
<td>$s^{-1}(C)$</td>
<td>set of states of the world that correspond to a state of nature in $C$, $s^{-1}(C) := {\omega \in Y : s(\omega) \in C}$</td>
<td>330</td>
</tr>
<tr>
<td>$S$</td>
<td>set of all vectors of pure strategies</td>
<td>77</td>
</tr>
<tr>
<td>$S$</td>
<td>set of states of nature in models of incomplete information</td>
<td>323</td>
</tr>
<tr>
<td>$S$</td>
<td>set of states of nature in a decision problem with experts</td>
<td>601</td>
</tr>
<tr>
<td>$S$</td>
<td>set of alternatives in a bargaining game</td>
<td>625</td>
</tr>
<tr>
<td>$S_i$</td>
<td>set of player $i$’s pure strategies</td>
<td>77</td>
</tr>
<tr>
<td>$S_h$</td>
<td>Shapley value</td>
<td>754</td>
</tr>
<tr>
<td>supp</td>
<td>support of a probability distribution</td>
<td>206</td>
</tr>
<tr>
<td>supp</td>
<td>support of a vector in $\mathbb{R}^n$</td>
<td>925</td>
</tr>
<tr>
<td>$t_i$</td>
<td>player $i$’s type in models of incomplete information</td>
<td>452</td>
</tr>
<tr>
<td>$T$</td>
<td>set of vectors of types in a Harsanyi model of incomplete information</td>
<td>347</td>
</tr>
<tr>
<td>$T$</td>
<td>number of stages in a finitely repeated game</td>
<td>528</td>
</tr>
<tr>
<td>$T_i$</td>
<td>player $i$’s type set in a Harsanyi model of incomplete information</td>
<td>347</td>
</tr>
<tr>
<td>$u$</td>
<td>payoff function in a strategic-form game</td>
<td>43, 601</td>
</tr>
<tr>
<td>$u_i$</td>
<td>player $i$’s utility function</td>
<td>14</td>
</tr>
<tr>
<td>$u_i$</td>
<td>player $i$’s payoff function</td>
<td>77</td>
</tr>
<tr>
<td>$u_i$</td>
<td>producer $i$’s production function in a market</td>
<td>703</td>
</tr>
<tr>
<td>$u_i'$</td>
<td>payoff of player $i$ at stage $t$ in a repeated game</td>
<td>527</td>
</tr>
<tr>
<td>$u'$</td>
<td>vector of payoffs at stage $t$ in a repeated game</td>
<td>527</td>
</tr>
<tr>
<td>$u(s)$</td>
<td>outcome of a game under strategy vector $s$</td>
<td>45</td>
</tr>
<tr>
<td>$U_i^j$</td>
<td>information set of player $i$ in an extensive-form game</td>
<td>54</td>
</tr>
<tr>
<td>$U_i$</td>
<td>mixed extension of player $i$’s payoff function</td>
<td>147</td>
</tr>
<tr>
<td>$U(C)$</td>
<td>uniform distribution over set $C$</td>
<td></td>
</tr>
<tr>
<td>$U[\alpha]$</td>
<td>scalar payoff function generated by projecting the payoffs in direction $\alpha$ in a game with payoff vectors</td>
<td>588</td>
</tr>
<tr>
<td>$v$</td>
<td>value of a two-player zero-sum game</td>
<td>114</td>
</tr>
<tr>
<td>$v$</td>
<td>coalitional function of a coalitional game</td>
<td>660</td>
</tr>
<tr>
<td>$v$</td>
<td>maxmin value of a two-player non-zero-sum game</td>
<td>113</td>
</tr>
<tr>
<td>$\bar{v}$</td>
<td>minmax value of a two-player non-zero-sum game</td>
<td>113</td>
</tr>
<tr>
<td>$v$</td>
<td>maximal private value of buyers in an auction</td>
<td>471</td>
</tr>
<tr>
<td>$v_0$</td>
<td>root of a game tree</td>
<td>42, 43</td>
</tr>
<tr>
<td>$v_i$</td>
<td>buyer $i$’s private value in an auction</td>
<td>91</td>
</tr>
<tr>
<td>$v^*$</td>
<td>superadditive closure of a coalitional game</td>
<td>732</td>
</tr>
<tr>
<td>$v_i$</td>
<td>player $i$’s maxmin value in a strategic-form game</td>
<td>103, 104, 176</td>
</tr>
<tr>
<td>$\bar{v}_i$</td>
<td>player $i$’s minmax value in a strategic-form game</td>
<td>177, 529</td>
</tr>
<tr>
<td>$\text{val}(A)$</td>
<td>value of a two-player zero-sum game whose payoff function is given by matrix $A$</td>
<td>588</td>
</tr>
<tr>
<td>$V$</td>
<td>set of edges in a graph</td>
<td>41, 43</td>
</tr>
<tr>
<td>$V$</td>
<td>set of individually rational payoffs in a repeated game</td>
<td>530</td>
</tr>
</tbody>
</table>
Notations

$V_0$ set of vertices in an extensive-form game where a chance move takes place 43
$V_i$ set of player $i$'s decision points in an extensive-form game 43
$V_i$ random variable representing buyer $i$'s private value in an auction 467
$\mathbb{V}$ buyer’s set of possible private values in a symmetric auction 471
$\mathbb{V}_i$ buyer $i$’s set of possible private values 466
$\mathbb{V}^N$ set of vectors of possible private values: $\mathbb{V}^N := \mathbb{V}_1 \times \mathbb{V}_2 \times \cdots \times \mathbb{V}_n$ 466
$w_i$ player $i$’s weight in a weighted majority game 664
$W_m$ collection of minimal winning coalitions in a simple monotonic game 826
$x_{-i}$ $x_{-i} := (x_j)_{j \neq i}$ 85
$x(S)$ $x(S) := \sum_{i \in S} x_i$, where $x \in \mathbb{R}^N$ 669
$X := \times_{i \in N} X_i$ 2
$X_k$ space of belief hierarchies of order $k$ 442
$X_{-i}$ $X_{-i} := \times_{j \neq i} X_j$ 85
$X(n)$ standard $(n - 1)$-dimensional simplex, $X(n) := \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = 1, \ x_i \geq 0 \ \forall i\}$ 935
$X(N; v)$ set of imputations in a coalitional game, $X(N; v) := \{x \in \mathbb{R}^n : x(N) = v(N), x_i \geq v(i) \ \forall i \in N\}$ 674, 802
$X^0(N; v)$ set of preimputations, $X^0(N; v) := \{x \in \mathbb{R}^n : x(N) = v(N)\}$ 805
$X(B; v)$ set of imputations for coalitional structure $B$, $X(B; v) := \{x \in \mathbb{R}^n : x(S) = v(S) \ \forall S \in B, x_i \geq v_i \ \forall i\}$ 674
$X^0(B; v)$ set of preimputations for coalitional structure $B$, $X^0(B; v) := \{x \in \mathbb{R}^n : x(S) = v(S) \ \forall S \in B\}$ 805
$Y$ set of states of the world 323, 334
$\tilde{\mathcal{Y}}(\omega)$ minimal belief subspace in state of the world $\omega$ 401
$\tilde{\mathcal{Y}}_i(\omega)$ minimal belief subspace of player $i$ in state of the world $\omega$ 403
$Z_k$ space of coherent belief hierarchies of order $k$ 445
$Z(P, Q; R)$ preference relation in which alternatives in $R$ are preferred to alternatives not in $R$, the preference over alternatives in $R$ is determined by $P$, and the preference over alternatives not in $R$ is determined by $Q$ 866
$Z(P^N, Q^N; R)$ preference profile in which the preference of individual $i$ is $Z(P_i, Q_i; R)$ 867
$\beta_i$ buyer $i$’s strategy in an auction 467
$\beta_i$ buyer $i$’s strategy in a selling mechanism 495
$\bar{\beta}_i$ buyer $i$’s strategy in a direct selling mechanism in which he reports his private value 495
$\Gamma$ extensive-form game 43, 50, 54
$\Gamma$ extension of a strategic-form game to mixed strategies 147
### Notations

- $\Gamma_T$: $T$-stage repeated game
- $\Gamma_\lambda$: discounted game with discount factor $\lambda$
- $\Gamma_\infty$: infinitely repeated game
- $\Gamma(x)$: subgame of an extensive-form game that starts at vertex $x$
- $\Gamma^*(p)$: extended game that includes a chance move that selects a vector of recommendations according to the probability distribution $p$ in the definition of a correlated equilibrium
- $\Delta(S)$: set of probability distributions over $S$
- $\varepsilon$: vector of constraints in the definition of perfect equilibrium
- $\varepsilon_i$: vector of constraints of player $i$ in the definition of perfect equilibrium
- $\varepsilon_i(s_i)$: minimal probability in which player $i$ selects pure strategy $s_i$ in the definition of perfect equilibrium
- $\theta(x)$: vector of excesses in decreasing order
- $\theta^k_i$: $A_k \approx [\theta^k_i(A_K), (1 - \theta^k_i)(A_D)]$
- $\lambda$: discount factor in a repeated game
- $\lambda_\alpha$: egalitarian solution with angle $\alpha$ of bargaining games
- $\mu_k$: belief hierarchy of order $k$
- $\chi_S$: incidence vector of a coalition
- $\Pi$: belief space: $\Pi = (Y, \mathcal{F}, s, (\pi_i)_{i \in N})$
- $\pi_i$: player $i$’s belief in a belief space
- $\sigma$: strategy in a decision problem with experts
- $\sigma_i$: mixed strategy of player $i$
- $\sigma_{-k}$: strategy of the player who is not player $k$ in a two-player game
- $\Sigma_i$: set of mixed strategies of player $i$
- $\tau_i$: strategy in a game with an outside observer $\Gamma^*(p)$
- $\tau_i^*$: player $i$’s strategy in a repeated game
- $\tau_i^*$: strategy in a game with an outside observer in which player $i$ follows the observer’s recommendation
- $\phi, \phi(S, d)$: solution concept for bargaining games
- $\phi$: solution concept for coalitional games
- $\psi$: solution concept for bankruptcy problems
- $\Omega$: universal belief space
**Introduction**

**What is game theory?**

Game theory is the name given to the methodology of using mathematical tools to model and analyze situations of interactive decision making. These are situations involving several decision makers (called *players*) with different goals, in which the decision of each affects the outcome for all the decision makers. This interactivity distinguishes game theory from standard decision theory, which involves a single decision maker, and it is its main focus. Game theory tries to predict the behavior of the players and sometimes also provides decision makers with suggestions regarding ways in which they can achieve their goals.

The foundations of game theory were laid down in the book *The Theory of Games and Economic Behavior*, published in 1944 by the mathematician John von Neumann and the economist Oskar Morgenstern. The theory has been developed extensively since then and today it has applications in a wide range of fields. The applicability of game theory is due to the fact that it is a context-free mathematical toolbox that can be used in any situation of interactive decision making. A partial list of fields where the theory is applied, along with examples of some questions that are studied within each field using game theory, includes:

- **Theoretical economics.** A market in which vendors sell items to buyers is an example of a game. Each vendor sets the price of the items that he or she wishes to sell, and each buyer decides from which vendor he or she will buy items and in what quantities. In models of markets, game theory attempts to predict the prices that will be set for the items along with the demand for each item, and to study the relationships between prices and demand. Another example of a game is an auction. Each participant in an auction determines the price that he or she will bid, with the item being sold to the highest bidder. In models of auctions, game theory is used to predict the bids submitted by the participants, the expected revenue of the seller, and how the expected revenue will change if a different auction method is used.

- **Networks.** The contemporary world is full of networks; the Internet and mobile telephone networks are two prominent examples. Each network user wishes to obtain the best possible service (for example, to send and receive the maximal amount of information in the shortest span of time over the Internet, or to conduct the highest-quality calls using a mobile telephone) at the lowest possible cost. A user has to choose an Internet service provider or a mobile telephone provider, where those providers are also players in the game, since they set the prices of the service they provide. Game theory tries to predict the behavior of all the participants in these markets. This game is more complicated from the perspective of the service providers than from the perspective
Introduction

of the buyers, because the service providers can cooperate with each other (for example, mobile telephone providers can use each other’s network infrastructure to carry communications in order to reduce costs), and game theory is used to predict which cooperative coalitions will be formed and suggests ways to determine a “fair” division of the profit of such cooperation among the participants.

- **Political science.** Political parties forming a governing coalition after parliamentary elections are playing a game whose outcome is the formation of a coalition that includes some of the parties. This coalition then divides government ministries and other elected offices, such as parliamentary speaker and committee chairmanships, among the members of the coalition. Game theory has developed indices measuring the power of each political party. These indices can predict or explain the division of government ministries and other elected offices given the results of the elections. Another branch of game theory suggests various voting methods and studies their properties.

- **Military applications.** A classical military application of game theory models a missile pursuing a fighter plane. What is the best missile pursuit strategy? What is the best strategy that the pilot of the plane can use to avoid being struck by the missile? Game theory has contributed to the field of defense the insight that the study of such situations requires strategic thinking: when coming to decide what you should do, put yourself in the place of your rival and think about what he/she would do and why, while taking into account that he/she is doing the same and knows that you are thinking strategically and that you are putting yourself in his/her place.

- **Inspection.** A broad family of problems from different fields can be described as two-player games in which one player is an entity that can profit by breaking the law and the other player is an “inspector” who monitors the behavior of the first player. One example of such a game is the activities of the International Atomic Energy Agency, in its role of enforcing the Treaty on the Non-Proliferation of Nuclear Weapons by inspecting the nuclear facilities of signatory countries. Additional examples include the enforcement of laws prohibiting drug smuggling, auditing of tax declarations by the tax authorities, and ticket inspections on public trains and buses.

- **Biology.** Plants and animals also play games. Evolution “determines” strategies that flowers use to attract insects for pollination and it “determines” strategies that the insects use to choose which flowers they will visit. Darwin’s principle of the “survival of the fittest” states that only those organisms with the inherited properties that are best adapted to the environmental conditions in which they are located will survive. This principle can be explained by the notion of *Evolutionarily Stable Strategy*, which is a variant of the notion of *Nash equilibrium*, the most prominent game-theoretic concept. The introduction of game theory to biology in general and to evolutionary biology in particular explains, sometimes surprisingly well, various biological phenomena.

Game theory has applications to other fields as well. For example, to philosophy it contributes some insights into concepts related to morality and social justice, and it raises questions regarding human behavior in various situations that are of interest to psychology. Methodologically, game theory is intimately tied to mathematics: the study of game-theoretic models makes use of a variety of mathematical tools, from probability and
combinatorics to differential equations and algebraic topology. Analyzing game-theoretic models sometimes requires developing new mathematical tools.

Traditionally, game theory is divided into two major subfields: strategic games, also called noncooperative games, and coalitional games, also called cooperative games. Broadly speaking, in strategic games the players act independently of each other, with each player trying to obtain the most desirable outcome given his or her preferences, while in coalitional games the same holds true with the stipulation that the players can agree on and sign binding contracts that enforce coordinated actions. Mechanisms enforcing such contracts include law courts and behavioral norms. Game theory does not deal with the quality or justification of these enforcement mechanisms; the cooperative game model simply assumes that such mechanisms exist and studies their consequences for the outcomes of the game.

The categories of strategic games and coalitional games are not well defined. In many cases interactive decision problems include aspects of both coalitional games and strategic games, and a complete theory of games should contain an amalgam of the elements of both types of models. Nevertheless, in a clear and focused introductory presentation of the main ideas of game theory it is convenient to stick to the traditional categorization. We will therefore present each of the two models, strategic games and coalitional games, separately. Chapters 1–14 are devoted to strategic games, and Chapters 15–20 are devoted to coalitional games. Chapters 21 and 22 are devoted to social choice and stable matching, which include aspects of both noncooperative and cooperative games.

How to use this book
The main objective of this book is to serve as an introductory textbook for the study of game theory at both the undergraduate and the graduate levels. A secondary goal is to serve as a reference book for students and scholars who are interested in an acquaintance with some basic or advanced topics of game theory. The number of introductory topics is large and different teachers may choose to teach different topics in introductory courses. We have therefore composed the book as a collection of chapters that are, to a large extent, independent of each other, enabling teachers to use any combination of the chapters as the basis for a course tailored to their individual taste. To help teachers plan a course, we have included an abstract at the beginning of each chapter that presents its content in a short and concise manner.

Each chapter begins with the basic concepts and eventually goes farther than what may be termed the “necessary minimum” in the subject that it covers. Most chapters include, in addition to introductory concepts, material that is appropriate for advanced courses. This gives teachers the option of teaching only the necessary minimum, presenting deeper material, or asking students to complement classroom lectures with independent readings or guided seminar presentations. We could not, of course, include all known results of game theory in one textbook, and therefore the end of each chapter contains references to other books and journal articles in which the interested reader can find more material for a deeper understanding of the subject. Each chapter also contains exercises, many of which are relatively easy, while some are more advanced and challenging.
This book was composed by mathematicians; the writing is therefore mathematically oriented, and every theorem in the book is presented with a proof. Nevertheless, an effort has been made to make the material clear and transparent, and every concept is illustrated with examples intended to impart as much intuition and motivation as possible. The book is appropriate for teaching undergraduate and graduate students in mathematics, computer science and exact sciences, economics and social sciences, engineering, and life sciences. It can be used as a textbook for teaching different courses in game theory, depending on the level of the students, the time available to the teacher, and the specific subject of the course. For example, it could be used in introductory level or advanced level semester courses on coalitional games, strategic games, a general course in game theory, or a course on applications of game theory. It could also be used for advanced mini-courses on, e.g., incomplete information (Chapters 9, 10, and 11), auctions (Chapter 12), or repeated games (Chapters 13 and 14). As mentioned previously, the material in the chapters of the book will in many cases encompass more than a teacher would choose to teach in a single course. This requires teachers to choose carefully which chapters to teach and which parts to cover in each chapter. For example, the material on strategic games (Chapters 4 and 5) can be taught without covering extensive-form games (Chapter 3) or utility theory (Chapter 2). Similarly, the material on games with incomplete information (Chapter 9) can be taught without teaching the other two chapters on models of incomplete information (Chapters 10 and 11).

For the sake of completeness, we have included an appendix containing the proofs of some theorems used throughout the book, including Brouwer’s Fixed Point Theorem, Kakutani’s Fixed Point Theorem, the Knaster–Kuratowski–Mazurkiewicz (KKM) Theorem, and the separating hyperplane theorem. The appendix also contains a brief survey of linear programming. A teacher can choose to prove each of these theorems in class, assign the proofs of the theorems as independent reading to the students, or state any of the theorems without proof based on the assumption that students will see the proofs in other courses.