Game Theory

Covering both noncooperative and cooperative games, this comprehensive introduction to game theory also includes some advanced chapters on auctions, games with incomplete information, games with vector payoffs, stable matchings, and the bargaining set. Mathematically oriented, the book presents every theorem alongside a proof. The material is presented clearly and every concept is illustrated with concrete examples from a broad range of disciplines. With numerous exercises the book is a thorough and extensive guide to game theory from undergraduate through graduate courses in economics, mathematics, computer science, engineering, and life sciences to being an authoritative reference for researchers.

MICHAEL MASCHLER was a professor in the Einstein Institute of Mathematics and the Center for the Study of Rationality at the Hebrew University of Jerusalem in Israel. He greatly contributed to cooperative game theory and to repeated games with incomplete information.

EILON SOLAN is a professor in the School of Mathematical Sciences at Tel Aviv University in Israel. The main topic of his research is repeated games. He serves on the editorial board of several academic journals.

SHMUEL ZAMIR is a professor emeritus in the Department of Statistics and the Center for the Study of Rationality at the Hebrew University of Jerusalem in Israel. The main topics of his research are games with incomplete information and auction theory. He is the editor-in-chief of the *International Journal of Game Theory*.

Game Theory

MICHAEL MASCHLER
EILON SOLAN
SHMUEL ZAMIR

Translated from Hebrew by Ziv Hellman

English Editor Mike Borns

CAMBRIDGE UNIVERSITY PRESS

CAMBRIDGE
UNIVERSITY PRESS

University Printing House, Cambridge CB2 8BS, United Kingdom

Cambridge University Press is part of the University of Cambridge.

It furthers the University's mission by disseminating knowledge in the pursuit of education, learning and research at the highest international levels of excellence.

www.cambridge.org
Information on this title: www.cambridge.org/9781107005488

First published 2013
4th printing 2015

Printed in the United Kingdom by Bell and Bain Ltd, Glasgow

A catalog record for this publication is available from the British Library

Library of Congress Cataloging in Publication data
Zamir, Shmuel.
[Torat ha-mishakim. English]
Game theory / Michael Maschler, Eilon Solan, Shmuel Zamir ; translated from Hebrew by Ziv Hellman ; English editor, Mike Borns.
pages cm
Translation of: Torat ha-mishakim / Shemu'el Zamir, Mikha'el Mashler ve-Elon Solan.
Includes bibliographical references and index.
ISBN 978-1-107-00548-8 (hardback)
1. Game theory. I. Maschler, Michael, 1927–2008. II. Solan, Eilon. III. Title.
QA269.Z3613 2013
519.3 – dc23 2012050827

ISBN 978-1-107-00548-8 Hardback

To Michael Maschler

Contents

Acknowledgments — *page* xiv
Notations — xv
Introduction — xxiii

1 The game of chess — 1

1.1 Schematic description of the game — 1
1.2 Analysis and results — 2
1.3 Remarks — 7
1.4 Exercises — 7

2 Utility theory — 9

2.1 Preference relations and their representation — 9
2.2 Preference relations over uncertain outcomes: the model — 12
2.3 The axioms of utility theory — 14
2.4 The characterization theorem for utility functions — 19
2.5 Utility functions and affine transformations — 22
2.6 Infinite outcome set — 23
2.7 Attitude towards risk — 23
2.8 Subjective probability — 26
2.9 Discussion — 27
2.10 Remarks — 31
2.11 Exercises — 31

3 Extensive-form games — 39

3.1 An example — 40
3.2 Graphs and trees — 41
3.3 Game trees — 42
3.4 Chomp: David Gale's game — 47
3.5 Games with chance moves — 49
3.6 Games with imperfect information — 52
3.7 Exercises — 57

4 Strategic-form games 75

4.1 Examples and definition of strategic-form games 76
4.2 The relationship between the extensive form and the strategic form 82
4.3 Strategic-form games: solution concepts 84
4.4 Notation 85
4.5 Domination 85
4.6 Second-price auctions 91
4.7 The order of elimination of dominated strategies 95
4.8 Stability: Nash equilibrium 95
4.9 Properties of the Nash equilibrium 100
4.10 Security: the maxmin concept 102
4.11 The effect of elimination of dominated strategies 106
4.12 Two-player zero-sum games 110
4.13 Games with perfect information 118
4.14 Games on the unit square 121
4.15 Remarks 128
4.16 Exercises 128

5 Mixed strategies 144

5.1 The mixed extension of a strategic-form game 145
5.2 Computing equilibria in mixed strategies 152
5.3 The proof of Nash's Theorem 166
5.4 Generalizing Nash's Theorem 170
5.5 Utility theory and mixed strategies 172
5.6 The maxmin and the minmax in n-player games 176
5.7 Imperfect information: the value of information 180
5.8 Evolutionarily stable strategies 186
5.9 Remarks 194
5.10 Exercises 194

6 Behavior strategies and Kuhn's Theorem 219

6.1 Behavior strategies 221
6.2 Kuhn's Theorem 226
6.3 Equilibria in behavior strategies 235
6.4 Kuhn's Theorem for infinite games 238
6.5 Remarks 243
6.6 Exercises 244

7 Equilibrium refinements 251

7.1 Subgame perfect equilibrium 252
7.2 Rationality, backward induction, and forward induction 260
7.3 Perfect equilibrium 262
7.4 Sequential equilibrium 271
7.5 Remarks 284
7.6 Exercises 284

8 Correlated equilibria 300

8.1 Examples 301
8.2 Definition and properties of correlated equilibrium 305
8.3 Remarks 313
8.4 Exercises 313

9 Games with incomplete information and common priors 319

9.1 The Aumann model of incomplete information and the concept of knowledge 322
9.2 The Aumann model of incomplete information with beliefs 334
9.3 An infinite set of states of the world 344
9.4 The Harsanyi model of games with incomplete information 345
9.5 Incomplete information as a possible interpretation of mixed strategies 361
9.6 The common prior assumption: inconsistent beliefs 365
9.7 Remarks 367
9.8 Exercises 368

10 Games with incomplete information: the general model 386

10.1 Belief spaces 386
10.2 Belief and knowledge 391
10.3 Examples of belief spaces 394
10.4 Belief subspaces 400
10.5 Games with incomplete information 407
10.6 The concept of consistency 415
10.7 Remarks 423
10.8 Exercises 423

11 The universal belief space 440

11.1 Belief hierarchies 442
11.2 Types 450

11.3 Definition of the universal belief space 453
11.4 Remarks 456
11.5 Exercises 456

12 Auctions 461

12.1 Notation 464
12.2 Common auction methods 464
12.3 Definition of a sealed-bid auction with private values 465
12.4 Equilibrium 468
12.5 The symmetric model with independent private values 471
12.6 The Envelope Theorem 484
12.7 Risk aversion 488
12.8 Mechanism design 492
12.9 Individually rational mechanisms 500
12.10 Finding the optimal mechanism 501
12.11 Remarks 508
12.12 Exercises 509

13 Repeated games 519

13.1 The model 520
13.2 Examples 521
13.3 The T-stage repeated game 524
13.4 Characterization of the set of equilibrium payoffs of the T-stage repeated game 530
13.5 Infinitely repeated games 537
13.6 The discounted game 542
13.7 Uniform equilibrium 546
13.8 Discussion 554
13.9 Remarks 555
13.10 Exercises 555

14 Repeated games with vector payoffs 569

14.1 Notation 570
14.2 The model 572
14.3 Examples 573
14.4 Connections between approachable and excludable sets 574
14.5 A geometric condition for the approachability of a set 576
14.6 Characterizations of convex approachable sets 585
14.7 Application 1: Repeated games with incomplete information 590
14.8 Application 2: Challenge the expert 600
14.9 Discussion 606
14.10 Remarks 607
14.11 Exercises 608

15 Bargaining games 622

15.1 Notation 625
15.2 The model 625
15.3 Properties of the Nash solution 626
15.4 Existence and uniqueness of the Nash solution 630
15.5 Another characterization of the Nash solution 635
15.6 The minimality of the properties of the Nash solution 639
15.7 Critiques of the properties of the Nash solution 641
15.8 Monotonicity properties 643
15.9 Bargaining games with more than two players 650
15.10 Remarks 653
15.11 Exercises 653

16 Coalitional games with transferable utility 659

16.1 Examples 661
16.2 Strategic equivalence 668
16.3 A game as a vector in a Euclidean space 670
16.4 Special families of games 671
16.5 Solution concepts 672
16.6 Geometric representation of the set of imputations 676
16.7 Remarks 678
16.8 Exercises 678

17 The core 686

17.1 Definition of the core 687
17.2 Balanced collections of coalitions 691
17.3 The Bondareva–Shapley Theorem 695
17.4 Market games 702
17.5 Additive games 712
17.6 The consistency property of the core 715
17.7 Convex games 717
17.8 Spanning tree games 721
17.9 Flow games 724
17.10 The core for general coalitional structures 732
17.11 Remarks 735
17.12 Exercises 735

18 The Shapley value 748

18.1 The Shapley properties 749
18.2 Solutions satisfying some of the Shapley properties 751
18.3 The definition and characterization of the Shapley value 754
18.4 Examples 758

Cambridge University Press
978-1-107-00548-8 - Game Theory
Michael Maschler, Eilon Solan and Shmuel Zamir
Frontmatter
More information

18.5 An alternative characterization of the Shapley value 760
18.6 Application: the Shapley–Shubik power index 763
18.7 Convex games 767
18.8 The consistency of the Shapley value 768
18.9 Remarks 774
18.10 Exercises 774

19 The bargaining set 782

19.1 Definition of the bargaining set 784
19.2 The bargaining set in two-player games 788
19.3 The bargaining set in three-player games 788
19.4 The bargaining set in convex games 794
19.5 Discussion 797
19.6 Remarks 798
19.7 Exercises 798

20 The nucleolus 801

20.1 Definition of the nucleolus 802
20.2 Nonemptiness and uniqueness of the nucleolus 805
20.3 Properties of the nucleolus 809
20.4 Computing the nucleolus 815
20.5 Characterizing the prenucleolus 816
20.6 The consistency of the nucleolus 823
20.7 Weighted majority games 825
20.8 The bankruptcy problem 831
20.9 Discussion 842
20.10 Remarks 843
20.11 Exercises 844

21 Social choice 853

21.1 Social welfare functions 856
21.2 Social choice functions 864
21.3 Non-manipulability 871
21.4 Discussion 873
21.5 Remarks 874
21.6 Exercises 874

22 Stable matching 884

22.1 The model 886
22.2 Existence of stable matching: the men's courtship algorithm 888
22.3 The women's courtship algorithm 890

Contents

22.4 Comparing matchings 892
22.5 Extensions 898
22.6 Remarks 905
22.7 Exercises 905

23 Appendices 916

23.1 Fixed point theorems 916
23.2 The Separating Hyperplane Theorem 943
23.3 Linear programming 945
23.4 Remarks 950
23.5 Exercises 950

References 958
Index 968

Acknowledgments

A great many people helped in the composition of the book and we are grateful to all of them. We thank Ziv Hellman, the devoted translator of the book. When he undertook this project he did not know that it would take up so much of his time. Nevertheless, he implemented all our requests with patience. We also thank Mike Borns, the English editor, who efficiently read through the text and brought it to its present state. We thank Ehud Lehrer who contributed exercises and answered questions that we had while writing the book, Uzi Motro who commented on the section on evolutionarily stable strategies, Dov Samet who commented on several chapters and contributed exercises, Tzachi Gilboa, Sergiu Hart, Aviad Heifetz, Bo'az Klartag, Vijay Krishna, Rida Laraki, Nimrod Megiddo, Abraham Neyman, Guni Orshan, Bezalel Peleg, David Schmeidler, Rann Smorodinsky, Peter Sudhölter, Yair Tauman, Rakesh Vohra, and Peter Wakker who answered our questions, and the many friends and students who read portions of the text, suggested improvements and exercises and spotted mistakes, including Alon Amit, Itai Arieli, Galit Ashkenazi-Golan, Yaron Azrieli, Shani Bar-Gera, Asaf Cohen, Ronen Eldan, Gadi Fibich, Tal Galili, Yuval Heller, John Levy, Maya Liran, C Maor, Ayala Mashiach-Yaakovi, Noa Nitzan, Gilad Pagi, Dori Reuveni, Eran Shmaya, Erez Sheiner, Omri Solan, Ron Solan, Roee Teper, Zorit Varmaz, and Saar Zilberman. Finally, we thank the Center for the Study of Rationality at the Hebrew University of Jerusalem and Hana Shemesh for the assistance they provided from the beginning of this project.

We thank Dr. Ron Peretz and his students, Prof. Krzysztof Apt and his students, Prof. Ehud Lehrer, Prof. Bezalel Peleg, Yotam Gafni, and Yonatan Elhanani for spotting typos in the first print of the book. These typos where corrected in this print.

Notations

The book makes use of large number of notations; we have striven to stick to accepted notation and to be consistent throughout the book. The coordinates of a vector are always denoted by a subscript index, $x = (x_i)_{i=1}^n$, while the indices of the elements of sequences are always denoted by a superscript index, $x^1, x^2, \ldots$ The index of a player in a set of players is always denoted by a subscript index, while a time index (in repeated games) is always denoted by a superscript index. The end of the proof of a theorem is indicated by $\square$, the end of an example is indicated by ◀, and the end of a remark is indicated by ♦.

For convenience we provide a list of the mathematical notation used throughout the book, accompanied by a short explanation and the pages on which they are formally defined. The notations that appear below are those that are used more than once.

0	chance move in an extensive-form game	50
$\vec{0}$	origin of a Euclidean space	570
$\emptyset$	strategy used by a player who has no decision vertices in an extensive-form game	5
$\mathbf{1}_A$	function that is equal to 1 on event A and to 0 otherwise	595
2^Y	collection of all subsets of Y	325
$\|X\|$	number of elements in finite set X	603
$\\|x\\|_\infty$	L_∞ norm, $\\|x\\|_\infty := \max_{i=1,2,\ldots,n} \|x_i\|$	531
$\\|x\\|$	norm of a vector, $\\|x\\| := \sqrt{\sum_{l=1}^d (x_l)^2}$	570
$A \vee B$	maximum matching (for men) in a matching problem	895
$A \wedge B$	maximum matching (for women) in a matching problem	896
$A \subseteq B$	set A contains set B or is equal to it	
$A \subset B$	set A strictly contains set B	
$\langle x, y \rangle$	inner product	570
$\langle\langle x^0, \ldots, x^k \rangle\rangle$	k-dimensional simplex	920
$\succsim_i$	preference relation of player i	14
$\succ_i$	strict preference relation of player i	10
$\approx_i$	indifference relation of player i	10, 897
$\succsim_P$	preference relation of an individual	857
$\succ_Q$	strict preference relation of society	857
$\approx_Q$	indifference relation of society	857
$x \geq y$	$x_k \geq y_k$ for each coordinate k, where x, y are vectors in a Euclidean space	625
$x > y$	$x \geq y$ and $x \neq y$	625

$x \gg y$	$x_k > y_k$ for each coordinate k, where x, y are vectors in a Euclidean space	625
$x + y$	sum of vectors in a Euclidean space, $(x + y)_k := x_k + y_k$	625
xy	coordinatewise product of vectors in a Euclidean space, $(xy)_k := x_k y_k$	625
$x + S$	$x + S := \{x + s : s \in S\}$, where $x \in \mathbb{R}^d$ and $S \subseteq \mathbb{R}^d$	625
xS	$xS := \{xs : s \in S\}$, where $x \in \mathbb{R}^d$ and $S \subseteq \mathbb{R}^d$	625
cx	product of real number c and vector x	625
cS	$cS := \{cs : s \in S\}$, where c is a real number and $S \subseteq \mathbb{R}^d$	625
$S + T$	sum of sets; $S + T := \{x + y : x \in S, y \in T\}$	625
$\lceil c \rceil$	smallest integer greater than or equal to c	534
$\lfloor c \rfloor$	largest integer less than or equal to c	534
$x^\top$	transpose of a vector, column vector that corresponds to row vector x	571
$\operatorname{argmax}_{x \in X} f(x)$	set of all x where function f attains its maximum in the set X	125, 625
$a(i)$	producer i's initial endowment in a market	703
A	set of actions in a decision problem with experts	601
A	set of alternatives	856
A_i	player i's action set in an extensive-form game, $A_i := \cup_{j=1}^{k_i} A(U_i^j)$	221
A_k	possible outcome of a game	13
$A(x)$	set of available actions at vertex x in an extensive-form game	44
$A(U_i)$	set of available actions at information set U_i of player i in an extensive-form game	54
b_i	buyer i's bid in an auction	91, 466
$b(S)$	$b(S) = \sum_{i \in S} b_i$ where $b \in \mathbb{R}^N$	669
$\mathrm{br}_{\mathrm{I}}(y)$	Player I's set of best replies to strategy y	125
$\mathrm{br}_{\mathrm{II}}(x)$	Player II's set of best replies to strategy x	125
B_i	player i's belief operator	392
B_i^p	set of states of the world in which the probability that player i ascribes to event E is at least p, $B_i^p(E) := \{\omega \in Y : \pi_i(E \mid \omega) \geq p\}$	426
$\mathrm{BZ}_i(N; v)$	Banzhaf value of a coalitional game	780
$\mathcal{B}$	coalitional structure	673
$\mathcal{B}_i^T$	set of behavior strategies of player i in a T-repeated game	525
$\mathcal{B}_i^\infty$	set of behavior strategies of player i in an infinitely repeated game	538
c	coalitional function of a cost game	661
c_+	maximum of c and 0	840
c_i	$c_i(v_i) := v_i - \frac{1-F_i(v_i)}{f_i(v_i)}$	501
C	function that dictates the amount that each buyer pays given the vector of bids in an auction	466

Cambridge University Press
978-1-107-00548-8 - Game Theory
Michael Maschler, Eilon Solan and Shmuel Zamir
Frontmatter
More information

Notations

$C(x)$	set of children of vertex x in an extensive-form game	5
$\mathcal{C}(N, v)$	core of a coalitional game	687
$\mathcal{C}(N, v; \mathcal{B})$	core for a coalitional structure	732
$\text{conv}\{x_1, \ldots, x_K\}$	smallest convex set that contains the vectors $\{x_1, \ldots, x_K\}$ Also called the convex hull of $\{x_1, \ldots, x_K\}$	530, 625, 917
d	disagreement point of a bargaining game	625
d_i	debt to creditor i in a bankruptcy problem	833
d^t	distance between average payoff and target set	581
$d(x, y)$	Euclidean distance between two vectors in Euclidean space	571
$d(x, S)$	Euclidean distance between point and set	571
$\mathcal{D}(\alpha, x)$	collection of coalitions whose excess is at least α, $\mathcal{D}(\alpha, x) := \{S \subseteq N, S \neq \emptyset: \quad e(S, x) \geq \alpha\}$	818
$e(S, x)$	excess of coalition S, $e(S, x) := v(S) - x(S)$	802
E	set of vertices of a graph	41, 43
E	estate of bankrupt entity in a bankruptcy problem	833
E	set of experts in a decision problem with experts	601
F	set of feasible payoffs in a repeated game	530, 578
F	social welfare function	857
F_i	cumulative distribution function of buyer i's private values in an auction	466
$F_i(\omega)$	atom of the partition $\mathcal{F}_i$ that contains ω	324
F^N	cumulative distribution function of joint distribution of vector of private values in an auction	466
$\mathcal{F}$	collection of all subgames in the game of chess	5
$\mathcal{F}$	family of bargaining games	625
$\mathcal{F}^N$	family of bargaining games with set of players N	650
$\mathcal{F}_d$	family of bargaining games in $\mathcal{F}$ where the set of alternatives is comprehensive and all alternatives are at least as good as the disagreement point, which is (0, 0)	644
$\mathcal{F}_i$	player i's information in an Aumann model of incomplete information	323
g^T	average payoff up to stage T (including) in a repeated game	572
G	graph	41
G	social choice function	865
h	history of a repeated game	525
h_t	history at stage t of a repeated game	602
$H(t)$	set of t-stage histories of a repeated game	525, 601
$H(\infty)$	set of plays in an infinitely repeated game	538
$H(\alpha, \beta)$	hyperplane, $H(\alpha, \beta) := \{x \in \mathbb{R}^d : \langle \alpha, x \rangle = \beta\}$	577, 943
$H^+(\alpha, \beta)$	half-space, $H^+(\alpha, \beta) := \{x \in \mathbb{R}^d : \langle \alpha, x \rangle \geq \beta\}$	577, 943
$H^-(\alpha, \beta)$	half-space, $H^-(\alpha, \beta) := \{x \in \mathbb{R}^d : \langle \alpha, x \rangle \leq \beta\}$	577, 943
i	player	
$-i$	set of all players except of player i	

Cambridge University Press
978-1-107-00548-8 - Game Theory
Michael Maschler, Eilon Solan and Shmuel Zamir
Frontmatter
More information

I	function that dictates the winner of an auction given the vector of bids	466
J	number of lotteries that compose a compound lottery	14
$J(x)$	player who chooses a move at vertex x of an extensive-form game	44
$-k$	player who is not k in a two-player game	571
k_i	number of information sets of player i in an extensive-form game	54
K	number of outcomes of a game	16
K_i	player i's knowledge operator	325
$\mathcal{KS}$, $\mathcal{KS}(S)$	Kalai–Smorodinsky solution to bargaining games	648
L	lottery: $L = [p_1(A_1), p_2(A_2), \ldots, p_K(A_K)]$	13
L	number of commodities in a market	703
$\widehat{L}$	compound lottery: $\widehat{L} = [q_1(L_1), \ldots, q_J(L_J)]$	14
$\mathcal{L}$	set of lotteries	13
$\widehat{\mathcal{L}}$	set of compound lotteries	15
$m(\epsilon)$	minimal coordinate of vector ε	264, 268
m_i	number of pure strategies of player i	147
$m_i(S)$	highest possible payoff to player i in a bargaining game	643
M	maximal absolute value of a payoff in a game	521
$M_{m,l}$	space of matrices of dimension $m \times l$	204
$M(\epsilon)$	maximal coordinate of vector ε	264, 268
$\mathcal{M}(N; v; \mathcal{B})$	bargaining set for coalitional structure $\mathcal{B}$	786
n	number of players	77
n	number of buyers in an auction	466
n_x	number of vertices in subgame $\Gamma(x)$	4
N	set of players	43, 833, 660
N	set of buyers in an auction	466
N	set of individuals	856
N	set of producers in a market	703
$\mathbb{N}$	set of natural numbers, $\mathbb{N} := \{1, 2, 3, \ldots\}$	
$\mathcal{N}$	$\mathcal{N}(S, d)$, Nash's solution to bargaining games	630
$\mathcal{N}(N; v)$	nucleolus of a coalitional game	805
$\mathcal{N}(N; v; \mathcal{B})$	nucleolus of a coalitional game for coalitional structure $\mathcal{B}$	805
$\mathcal{N}(N; v; K)$	nucleolus relative to set K	804
O	set of outcomes	13, 43
p	common prior in a Harsanyi game with incomplete information	347
p_k	probability that the outcome of lottery L is A_k	13
p_x	probability distribution over actions at chance move x	50
P	binary relation	857

Notations

P	set of all weakly balancing weights for collection $\mathcal{D}^*$ of all coalitions	701
$\mathbf{P}$	common prior in an Aumann model of incomplete information	334
$\mathbf{P}_\sigma(x)$	probability that the play reaches vertex x when the players implement strategy vector σ in an extensive-form game	254
$\mathbf{P}_\sigma(U)$	probability that the play reaches a vertex in information set U when the players implement strategy vector σ in an extensive-form game	273
P^N	vector of preference relations	857
$PO(S)$	set of efficient (Pareto optimal) points in S	627
$PO^W(S)$	set of weakly efficient points in S	627
$\mathcal{P}(A)$	set of all strict preference relations over a set of alternatives A	857
$\mathcal{P}(N)$	collection of nonempty subsets of N, $\mathcal{P}(N) := \{S \subseteq N, S \neq \emptyset\}$	670, 701
$\mathcal{P}^*(A)$	set of all preference relations over a set of alternatives A	857
$\mathcal{PN}(N; v)$	prenucleolus of a coalitional game	805
$\mathcal{PN}(N; v; \mathcal{B})$	prenucleolus of a coalitional game for coalitional structure $\mathcal{B}$	805
q	quota in a weighted majority game	664
$q(w)$	minimal weight of a winning coalition in a weighted majority game, $q(w) := \min_{S \in \mathcal{W}^m} w(S)$	828
$\mathbb{Q}_{++}$	set of positive rational numbers	
r_k	total probability that the result of a compound lottery is A_k	18
$R_1(p)$	set of possible payoffs when Player 1 plays mixed action p, $R_1(p) := \{puq^\top : q \in \Delta(\mathcal{J})\}$	576
$R_2(p)$	set of possible payoffs when Player 2 plays mixed action q, $R_2(p) := \{puq^\top : q \in \Delta(\mathcal{I})\}$	576
$\mathbb{R}$	real line	
$\mathbb{R}_+$	set of nonnegative numbers	
$\mathbb{R}_{++}$	set of positive numbers	
$\mathbb{R}^n$	n-dimensional Euclidean space	
$\mathbb{R}^n_+$	nonnegative orthant in an n-dimensional Euclidean space, $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_i \geq 0, \quad \forall i = 1, 2, \ldots, n\}$	
$\mathbb{R}^S$	$\|S\|$-dimensional Euclidean space, where each coordinate corresponds to a player in S	669
range(G)	range of a social choice function	870
s	strategy vector	45
$\mathfrak{s}$	function that assigns a state of nature to each state of the world	323
s^t	action vector played at stage t of a repeated game	525
s_i	strategy of player i	45, 56

Cambridge University Press
978-1-107-00548-8 - Game Theory
Michael Maschler, Eilon Solan and Shmuel Zamir
Frontmatter
More information

s_t	state of nature that corresponds to type vector t in a Harsanyi game with incomplete information	347
$\mathfrak{s}^{-1}(C)$	set of states of the world that correspond to a state of nature in C, $\mathfrak{s}^{-1}(C) := \{\omega \in Y : s(\omega) \in C\}$	330
S	set of all vectors of pure strategies	77
S	set of states of nature in models of incomplete information	323
S	set of states of nature in a decision problem with experts	601
S	set of alternatives in a bargaining game	625
S_i	set of player i's pure strategies	77
Sh	Shapley value	754
supp	support of a probability distribution	206
supp	support of a vector in $\mathbb{R}^n$	925
t_i	player i's type in models of incomplete information	452
T	set of vectors of types in a Harsanyi model of incomplete information	347
T	number of stages in a finitely repeated game	528
T_i	player i's type set in a Harsanyi model of incomplete information	347
u	payoff function in a strategic-form game	43, 601
u_i	player i's utility function	14
u_i	player i's payoff function	77
u_i	producer i's production function in a market	703
u_t^i	payoff of player i at stage t in a repeated game	527
u^t	vector of payoffs at stage t in a repeated game	527
$u(s)$	outcome of a game under strategy vector s	45
U_i^j	information set of player i in an extensive-form game	54
U_i	mixed extension of player i's payoff function	147
$U(C)$	uniform distribution over set C	
$U[\alpha]$	scalar payoff function generated by projecting the payoffs in direction α in a game with payoff vectors	588
v	value of a two-player zero-sum game	114
v	coalitional function of a coalitional game	660
$\underline{v}$	maxmin value of a two-player non-zero-sum game	113
$\overline{v}$	minmax value of a two-player non-zero-sum game	113
$\overline{v}$	maximal private value of buyers in an auction	471
v_0	root of a game tree	42, 43
v_i	buyer i's private value in an auction	91
v^*	superadditive closure of a coalitional game	732
$\underline{v}_i$	player i's maxmin value in a strategic-form game	103, 104, 176
$\overline{v}_i$	player i's minmax value in a strategic-form game	177, 529
val(A)	value of a two-player zero-sum game whose payoff function is given by matrix A	588
V	set of edges in a graph	41, 43
V	set of individually rational payoffs in a repeated game	530

Cambridge University Press
978-1-107-00548-8 - Game Theory
Michael Maschler, Eilon Solan and Shmuel Zamir
Frontmatter
More information

Notations

V_0	set of vertices in an extensive-form game where a chance move takes place	43
V_i	set of player i's decision points in an extensive-form game	43
V_i	random variable representing buyer i's private value in an auction	467
$\mathbb{V}$	buyer's set of possible private values in a symmetric auction	471
$\mathbb{V}_i$	buyer i's set of possible private values	466
$\mathbb{V}^N$	set of vectors of possible private values: $\mathbb{V}^N := \mathbb{V}_1 \times \mathbb{V}_2 \times \cdots \times \mathbb{V}_n$	466
w_i	player i's weight in a weighted majority game	664
$\mathcal{W}^m$	collection of minimal winning coalitions in a simple monotonic game	826
x_{-i}	$x_{-i} := (x_j)_{j \neq i}$	85
$x(S)$	$x(S) := \sum_{i \in S} x_i$, where $x \in \mathbb{R}^N$	669
X	$X := \times_{i \in N} X_i$	2
X_k	space of belief hierarchies of order k	442
X_{-i}	$X_{-i} := \times_{j \neq i} X_j$	85
$X(n)$	standard $(n-1)$-dimensional simplex, $X(n) := \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i = 1, \quad x_i \geq 0 \quad \forall i\}$	935
$X(N; v)$	set of imputations in a coalitional game, $X(N; v) := \{x \in \mathbb{R}^n : x(N) = v(N), x_i \geq v(i) \quad \forall i \in N\}$	674, 802
$X^0(N; v)$	set of preimputations, $X^0(N; v) := \{x \in \mathbb{R}^N : x(N) = v(N)\}$	805
$X(\mathcal{B}; v)$	set of imputations for coalitional structure $\mathcal{B}$, $X(\mathcal{B}; v) := \{x \in \mathbb{R}^N : x(S) = v(S) \quad \forall S \in \mathcal{B}, x_i \geq v_i \quad \forall i\}$	674
$X^0(\mathcal{B}; v)$	set of preimputations for coalitional structure $\mathcal{B}$, $X^0(\mathcal{B}; v) := \{x \in \mathbb{R}^N : x(S) = v(S) \quad \forall S \in \mathcal{B}\}$	805
Y	set of states of the world	323, 334
$\widetilde{Y}(\omega)$	minimal belief subspace in state of the world ω	401
$\widetilde{Y}_i(\omega)$	minimal belief subspace of player i in state of the world ω	403
Z_k	space of coherent belief hierarchies of order k	445
$Z(P, Q; R)$	preference relation in which alternatives in R are preferred to alternatives not in R, the preference over alternatives in R is determined by P, and the preference over alternatives not in R is determined by Q	866
$Z(P^N, Q^N; R)$	preference profile in which the preference of individual i is $Z(P_i, Q_i; R)$	867
β_i	buyer i's strategy in an auction	467
β_i	buyer i's strategy in a selling mechanism	495
β_i^*	buyer i's strategy in a direct selling mechanism in which he reports his private value	495
Γ	extensive-form game	43, 50, 54
Γ	extension of a strategic-form game to mixed strategies	147

Γ_T	T-stage repeated game	528
Γ_λ	discounted game with discount factor λ	544
Γ_∞	infinitely repeated game	539
$\Gamma(x)$	subgame of an extensive-form game that starts at vertex x	4, 45, 55
$\Gamma^*(p)$	extended game that includes a chance move that selects a vector of recommendations according to the probability distribution p in the definition of a correlated equilibrium	305
$\Delta(S)$	set of probability distributions over S	146
ε	vector of constraints in the definition of perfect equilibrium	264
ε_i	vector of constraints of player i in the definition of perfect equilibrium	264
$\varepsilon_i(s_i)$	minimal probability in which player i selects pure strategy s_i in the definition of perfect equilibrium	264
$\theta(x)$	vector of excesses in decreasing order	802
θ_i^k	$A_k \approx [\theta_i^k(A_K), (1-\theta_i^k)(A_0)]$	20
λ	discount factor in a repeated game	543
λ_α	egalitarian solution with angle α of bargaining games	640
μ^k	belief hierarchy of order k	442
χ^S	incidence vector of a coalition	693
Π	belief space: $\Pi = (Y, \mathcal{F}, s, (\pi_i)_{i\in N})$	466
π_i	player i's belief in a belief space	387
σ	strategy in a decision problem with experts	601
σ_i	mixed strategy of player i	146
σ_{-k}	strategy of the player who is not player k in a two-player game	571
Σ_i	set of mixed strategies of player i	147
τ_i	strategy in a game with an outside observer $\Gamma^*(p)$	305
τ_i	player i's strategy in a repeated game	525, 538
τ_i^*	strategy in a game with an outside observer in which player i follows the observer's recommendation	306
$\varphi, \varphi(S,d)$	solution concept for bargaining games	626
φ	solution concept for coalitional games	673
φ	solution concept for bankruptcy problems	833
Ω	universal belief space	453

Introduction

What is game theory?

Game theory is the name given to the methodology of using mathematical tools to model and analyze situations of interactive decision making. These are situations involving several decision makers (called *players*) with different goals, in which the decision of each affects the outcome for all the decision makers. This interactivity distinguishes game theory from standard decision theory, which involves a single decision maker, and it is its main focus. Game theory tries to predict the behavior of the players and sometimes also provides decision makers with suggestions regarding ways in which they can achieve their goals.

The foundations of game theory were laid down in the book *The Theory of Games and Economic Behavior*, published in 1944 by the mathematician John von Neumann and the economist Oskar Morgenstern. The theory has been developed extensively since then and today it has applications in a wide range of fields. The applicability of game theory is due to the fact that it is a context-free mathematical toolbox that can be used in any situation of interactive decision making. A partial list of fields where the theory is applied, along with examples of some questions that are studied within each field using game theory, includes:

- **Theoretical economics.** A market in which vendors sell items to buyers is an example of a game. Each vendor sets the price of the items that he or she wishes to sell, and each buyer decides from which vendor he or she will buy items and in what quantities. In models of markets, game theory attempts to predict the prices that will be set for the items along with the demand for each item, and to study the relationships between prices and demand. Another example of a game is an auction. Each participant in an auction determines the price that he or she will bid, with the item being sold to the highest bidder. In models of auctions, game theory is used to predict the bids submitted by the participants, the expected revenue of the seller, and how the expected revenue will change if a different auction method is used.
- **Networks.** The contemporary world is full of networks; the Internet and mobile telephone networks are two prominent examples. Each network user wishes to obtain the best possible service (for example, to send and receive the maximal amount of information in the shortest span of time over the Internet, or to conduct the highest-quality calls using a mobile telephone) at the lowest possible cost. A user has to choose an Internet service provider or a mobile telephone provider, where those providers are also players in the game, since they set the prices of the service they provide. Game theory tries to predict the behavior of all the participants in these markets. This game is more complicated from the perspective of the service providers than from the perspective

of the buyers, because the service providers can cooperate with each other (for example, mobile telephone providers can use each other's network infrastructure to carry communications in order to reduce costs), and game theory is used to predict which cooperative coalitions will be formed and suggests ways to determine a "fair" division of the profit of such cooperation among the participants.

- **Political science.** Political parties forming a governing coalition after parliamentary elections are playing a game whose outcome is the formation of a coalition that includes some of the parties. This coalition then divides government ministries and other elected offices, such as parliamentary speaker and committee chairmanships, among the members of the coalition. Game theory has developed indices measuring the power of each political party. These indices can predict or explain the division of government ministries and other elected offices given the results of the elections. Another branch of game theory suggests various voting methods and studies their properties.
- **Military applications.** A classical military application of game theory models a missile pursuing a fighter plane. What is the best missile pursuit strategy? What is the best strategy that the pilot of the plane can use to avoid being struck by the missile? Game theory has contributed to the field of defense the insight that the study of such situations requires strategic thinking: when coming to decide what you should do, put yourself in the place of your rival and think about what he/she would do and why, while taking into account that he/she is doing the same and knows that you are thinking strategically and that you are putting yourself in his/her place.
- **Inspection.** A broad family of problems from different fields can be described as two-player games in which one player is an entity that can profit by breaking the law and the other player is an "inspector" who monitors the behavior of the first player. One example of such a game is the activities of the International Atomic Energy Agency, in its role of enforcing the Treaty on the Non-Proliferation of Nuclear Weapons by inspecting the nuclear facilities of signatory countries. Additional examples include the enforcement of laws prohibiting drug smuggling, auditing of tax declarations by the tax authorities, and ticket inspections on public trains and buses.
- **Biology.** Plants and animals also play games. Evolution "determines" strategies that flowers use to attract insects for pollination and it "determines" strategies that the insects use to choose which flowers they will visit. Darwin's principle of the "survival of the fittest" states that only those organisms with the inherited properties that are best adapted to the environmental conditions in which they are located will survive. This principle can be explained by the notion of *Evolutionarily Stable Strategy*, which is a variant of the notion of *Nash equilibrium*, the most prominent game-theoretic concept. The introduction of game theory to biology in general and to evolutionary biology in particular explains, sometimes surprisingly well, various biological phenomena.

Game theory has applications to other fields as well. For example, to philosophy it contributes some insights into concepts related to morality and social justice, and it raises questions regarding human behavior in various situations that are of interest to psychology. Methodologically, game theory is intimately tied to mathematics: the study of game-theoretic models makes use of a variety of mathematical tools, from probability and combinatorics to differential equations and algebraic topology. Analyzing game-theoretic models sometimes requires developing new mathematical tools.

Traditionally, game theory is divided into two major subfields: strategic games, also called noncooperative games, and coalitional games, also called cooperative games. Broadly speaking, in strategic games the players act independently of each other, with each player trying to obtain the most desirable outcome given his or her preferences, while in coalitional games the same holds true with the stipulation that the players can agree on and sign binding contracts that enforce coordinated actions. Mechanisms enforcing such contracts include law courts and behavioral norms. Game theory does not deal with the quality or justification of these enforcement mechanisms; the cooperative game model simply assumes that such mechanisms exist and studies their consequences for the outcomes of the game.

The categories of strategic games and coalitional games are not well defined. In many cases interactive decision problems include aspects of both coalitional games and strategic games, and a complete theory of games should contain an amalgam of the elements of both types of models. Nevertheless, in a clear and focused introductory presentation of the main ideas of game theory it is convenient to stick to the traditional categorization. We will therefore present each of the two models, strategic games and coalitional games, separately. Chapters 1–14 are devoted to strategic games, and Chapters 15–20 are devoted to coalitional games. Chapters 21 and 22 are devoted to social choice and stable matching, which include aspects of both noncooperative and cooperative games.

How to use this book

The main objective of this book is to serve as an introductory textbook for the study of game theory at both the undergraduate and the graduate levels. A secondary goal is to serve as a reference book for students and scholars who are interested in an acquaintance with some basic or advanced topics of game theory. The number of introductory topics is large and different teachers may choose to teach different topics in introductory courses. We have therefore composed the book as a collection of chapters that are, to a large extent, independent of each other, enabling teachers to use any combination of the chapters as the basis for a course tailored to their individual taste. To help teachers plan a course, we have included an abstract at the beginning of each chapter that presents its content in a short and concise manner.

Each chapter begins with the basic concepts and eventually goes farther than what may be termed the “necessary minimum” in the subject that it covers. Most chapters include, in addition to introductory concepts, material that is appropriate for advanced courses. This gives teachers the option of teaching only the necessary minimum, presenting deeper material, or asking students to complement classroom lectures with independent readings or guided seminar presentations. We could not, of course, include all known results of game theory in one textbook, and therefore the end of each chapter contains references to other books and journal articles in which the interested reader can find more material for a deeper understanding of the subject. Each chapter also contains exercises, many of which are relatively easy, while some are more advanced and challenging.

This book was composed by mathematicians; the writing is therefore mathematically oriented, and every theorem in the book is presented with a proof. Nevertheless, an effort has been made to make the material clear and transparent, and every concept is illustrated with examples intended to impart as much intuition and motivation as possible. The book is appropriate for teaching undergraduate and graduate students in mathematics, computer science and exact sciences, economics and social sciences, engineering, and life sciences. It can be used as a textbook for teaching different courses in game theory, depending on the level of the students, the time available to the teacher, and the specific subject of the course. For example, it could be used in introductory level or advanced level semester courses on coalitional games, strategic games, a general course in game theory, or a course on applications of game theory. It could also be used for advanced mini-courses on, e.g., incomplete information (Chapters 9, 10, and 11), auctions (Chapter 12), or repeated games (Chapters 13 and 14). As mentioned previously, the material in the chapters of the book will in many cases encompass more than a teacher would choose to teach in a single course. This requires teachers to choose carefully which chapters to teach and which parts to cover in each chapter. For example, the material on strategic games (Chapters 4 and 5) can be taught without covering extensive-form games (Chapter 3) or utility theory (Chapter 2). Similarly, the material on games with incomplete information (Chapter 9) can be taught without teaching the other two chapters on models of incomplete information (Chapters 10 and 11).

For the sake of completeness, we have included an appendix containing the proofs of some theorems used throughout the book, including Brouwer's Fixed Point Theorem, Kakutani's Fixed Point Theorem, the Knaster–Kuratowski–Mazurkiewicz (KKM) Theorem, and the separating hyperplane theorem. The appendix also contains a brief survey of linear programming. A teacher can choose to prove each of these theorems in class, assign the proofs of the theorems as independent reading to the students, or state any of the theorems without proof based on the assumption that students will see the proofs in other courses.