Physical Mathematics

Unique in its clarity, examples, and range, Physical Mathematics explains as simply as possible the mathematics that graduate students and professional physicists need in their courses and research. The author illustrates the mathematics with numerous physical examples drawn from contemporary research. In addition to basic subjects such as linear algebra, Fourier analysis, complex variables, differential equations, and Bessel functions, this textbook covers topics such as the singular-value decomposition, Lie algebras, the tensors and forms of general relativity, the central limit theorem and Kolmogorov test of statistics, the Monte Carlo methods of experimental and theoretical physics, the renormalization group of condensed-matter physics, and the functional derivatives and Feynman path integrals of quantum field theory. Solutions to exercises are available for instructors at www.cambridge.org/cahill.

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Physical Mathematics

KEVIN CAHILL

University of New Mexico
For Ginette, Mike, Sean, Peter, Mia, and James,
and in honor of Muntadhar al-Zaidi.
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Preface

To the students: you will find some physics crammed in amongst the mathematics. Don’t let the physics bother you. As you study the math, you’ll learn some physics without extra effort. The physics is a freebie. I have tried to explain the math you need for physics and have left out the rest.

To the professors: the book is for students who also are taking mechanics, electrodynamics, quantum mechanics, and statistical mechanics nearly simultaneously and who soon may use probability or path integrals in their research. Linear algebra and Fourier analysis are the keys to physics, so the book starts with them, but you may prefer to skip the algebra or postpone the Fourier analysis. The book is intended to support a one- or two-semester course for graduate students or advanced undergraduates. The first seven, eight, or nine chapters fit in one semester, the others in a second. A list of errata is maintained at panda.unm.edu/cahill, and solutions to all the exercises are available for instructors at www.cambridge.org/cahill.

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