METHODS OF APPLIED MATHEMATICS FOR ENGINEERS AND SCIENTISTS

Based on course notes from more than twenty years of teaching engineering and physical sciences at Michigan Technological University, Tomas B. Co’s engineering mathematics textbook is rich with examples, applications, and exercises. Professor Co uses analytical approaches to solve smaller problems to provide mathematical insight and understanding, and numerical methods for large and complex problems. The book emphasizes applying matrices with strong attention to matrix structure and computational issues such as sparsity and efficiency. Chapters on vector calculus and integral theorems are used to build coordinate-free physical models, with special emphasis on orthogonal coordinates. Chapters on ordinary differential equations and partial differential equations cover both analytical and numerical approaches. Topics on analytical solutions include similarity transform methods, direct formulas for series solutions, bifurcation analysis, Lagrange-Charpit formulas, and shocks/rarefaction. Topics on numerical methods include stability analysis, differential algebraic equations, high-order finite-difference formulas, and Delaunay meshes. MATLAB implementations of the methods and concepts are fully integrated.

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Methods of Applied Mathematics for Engineers and Scientists

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# Contents

## Preface

**Contents**

## 1 MATRIX THEORY

1 **Matrix Algebra** .................................................. 3  
   1.1 Definitions and Notations 4  
   1.2 Fundamental Matrix Operations 6  
   1.3 Properties of Matrix Operations 18  
   1.4 Block Matrix Operations 30  
   1.5 Matrix Calculus 31  
   1.6 Sparse Matrices 39  
   1.7 Exercises 41

2 **Solution of Multiple Equations** ........................... 54  
   2.1 Gauss-Jordan Elimination 55  
   2.2 LU Decomposition 59  
   2.3 Direct Matrix Splitting 65  
   2.4 Iterative Solution Methods 66  
   2.5 Least-Squares Solution 71  
   2.6 QR Decomposition 77  
   2.7 Conjugate Gradient Method 78  
   2.8 GMRES 79  
   2.9 Newton’s Method 80  
   2.10 Enhanced Newton Methods via Line Search 82  
   2.11 Exercises 86

3 **Matrix Analysis** ..................................... 99  
   3.1 Matrix Operators 100  
   3.2 Eigenvalues and Eigenvectors 107  
   3.3 Properties of Eigenvalues and Eigenvectors 113  
   3.4 Schur Triangularization and Normal Matrices 116  
   3.5 Diagonalization 117  
   3.6 Jordan Canonical Form 118  
   3.7 Functions of Square Matrices 120
3.8 Stability of Matrix Operators 124
3.9 Singular Value Decomposition 127
3.10 Polar Decomposition 132
3.11 Matrix Norms 135
3.12 Exercises 138

II VECTORS AND TENSORS

4 Vector and Tensor Algebra and Calculus ............................................. 149
  4.1 Notations and Fundamental Operations 150
  4.2 Vector Algebra Based on Orthonormal Basis Vectors 154
  4.3 Tensor Algebra 157
  4.4 Matrix Representation of Vectors and Tensors 162
  4.5 Differential Operations for Vector Functions of One Variable 164
  4.6 Application to Position Vectors 165
  4.7 Differential Operations for Vector Fields 169
  4.8 Curvilinear Coordinate System: Cylindrical and Spherical 184
  4.9 Orthogonal Curvilinear Coordinates 189
  4.10 Exercises 196

5 Vector Integral Theorems ................................................................. 204
  5.1 Green’s Lemma 205
  5.2 Divergence Theorem 208
  5.3 Stokes’ Theorem and Path Independence 210
  5.4 Applications 215
  5.5 Leibnitz Derivative Formula 224
  5.6 Exercises 225

III ORDINARY DIFFERENTIAL EQUATIONS

6 Analytical Solutions of Ordinary Differential Equations ............... 235
  6.1 First-Order Ordinary Differential Equations 236
  6.2 Separable Forms via Similarity Transformations 238
  6.3 Exact Differential Equations via Integrating Factors 242
  6.4 Second-Order Ordinary Differential Equations 245
  6.5 Multiple Differential Equations 250
  6.6 Decoupled System Descriptions via Diagonalization 258
  6.7 Laplace Transform Methods 262
  6.8 Exercises 263

7 Numerical Solution of Initial and Boundary Value Problems .......... 273
  7.1 Euler Methods 274
  7.2 Runge Kutta Methods 276
  7.3 Multistep Methods 282
  7.4 Difference Equations and Stability 291
  7.5 Boundary Value Problems 299
  7.6 Differential Algebraic Equations 303
  7.7 Exercises 305
8 Qualitative Analysis of Ordinary Differential Equations ............................ 311
8.1 Existence and Uniqueness ......................................................... 312
8.2 Autonomous Systems and Equilibrium Points .......................... 313
8.3 Integral Curves, Phase Space, Flows, and Trajectories .............. 314
8.4 Lyapunov and Asymptotic Stability ........................................... 317
8.5 Phase-Plane Analysis of Linear Second-Order
    Autonomous Systems ..................................................................... 321
8.6 Linearization Around Equilibrium Points ................................. 327
8.7 Method of Lyapunov Functions .................................................. 330
8.8 Limit Cycles ........................................................................... 332
8.9 Bifurcation Analysis ................................................................. 340
8.10 Exercises ................................................................................ 340

9 Series Solutions of Linear Ordinary Differential Equations ............. 347
9.1 Power Series Solutions ............................................................... 347
9.2 Legendre Equations .................................................................. 358
9.3 Bessel Equations .................................................................... 363
9.4 Properties and Identities of Bessel Functions and
    Modified Bessel Functions ......................................................... 369
9.5 Exercises ................................................................................ 371

IV PARTIAL DIFFERENTIAL EQUATIONS

10 First-Order Partial Differential Equations and the Method of
    Characteristics .................................................................................. 379
10.1 The Method of Characteristics ................................................... 380
10.2 Alternate Forms and General Solutions ....................................... 387
10.3 The Lagrange-Charpit Method .................................................... 389
10.4 Classification Based on Principal Parts ....................................... 393
10.5 Hyperbolic Systems of Equations ................................................. 397
10.6 Exercises ................................................................................ 399

11 Linear Partial Differential Equations ............................................. 405
11.1 Linear Partial Differential Operator .............................................. 406
11.2 Reducible Linear Partial Differential Equations .......................... 408
11.3 Method of Separation of Variables ............................................. 411
11.4 Nonhomogeneous Partial Differential Equations ................. 431
11.5 Similarity Transformations ......................................................... 439
11.6 Exercises ................................................................................ 443

12 Integral Transform Methods ................................................................. 450
12.1 General Integral Transforms ....................................................... 451
12.2 Fourier Transforms ................................................................. 452
12.3 Solution of PDEs Using Fourier Transforms ................................ 459
12.4 Laplace Transforms ................................................................. 464
12.5 Solution of PDEs Using Laplace Transforms ........................... 474
12.6 Method of Images .................................................................. 476
12.7 Exercises ................................................................................ 477
Contents

13 Finite Difference Methods ............................................. 483
   13.1 Finite Difference Approximations 484
   13.2 Time-Independent Equations 491
   13.3 Time-Dependent Equations 504
   13.4 Stability Analysis 512
   13.5 Exercises 519

14 Method of Finite Elements ............................................. 523
   14.1 The Weak Form 524
   14.2 Triangular Finite Elements 527
   14.3 Assembly of Finite Elements 533
   14.4 Mesh Generation 539
   14.5 Summary of Finite Element Method 541
   14.6 Axisymmetric Case 546
   14.7 Time-Dependent Systems 547
   14.8 Exercises 552

Bibliography

Index  B-1

A Additional Details and Fortification for Chapter 1 ................. 561
   A.1 Matrix Classes and Special Matrices 561
   A.2 Motivation for Matrix Operations from Solution of Equations 568
   A.3 Taylor Series Expansion 572
   A.4 Proofs for Lemma and Theorems of Chapter 1 576
   A.5 Positive Definite Matrices 586

B Additional Details and Fortification for Chapter 2 ............... 589
   B.1 Gauss Jordan Elimination Algorithm 589
   B.2 SVD to Determine Gauss-Jordan Matrices Q and W 594
   B.3 Boolean Matrices and Reducible Matrices 595
   B.4 Reduction of Matrix Bandwidth 600
   B.5 Block LU Decomposition 602
   B.6 Matrix Splitting: Diakoptic Method and Schur Complement Method 605
   B.7 Linear Vector Algebra: Fundamental Concepts 611
   B.8 Determination of Linear Independence of Functions 614
   B.9 Gram-Schmidt Orthogonalization 616
   B.10 Proofs for Lemma and Theorems in Chapter 2 617
   B.11 Conjugate Gradient Algorithm 620
   B.12 GMRES Algorithm 629
   B.13 Enhanced-Newton Using Double-Dogleg Method 635
   B.14 Nonlinear Least Squares via Levenberg-Marquardt 639

C Additional Details and Fortification for Chapter 3 ............... 644
   C.1 Proofs of Lemmas and Theorems of Chapter 3 644
   C.2 QR Method for Eigenvalue Calculations 649
   C.3 Calculations for the Jordan Decomposition 655
## Contents

C.4 Schur Triangularization and SVD 658  
C.5 Sylvester’s Matrix Theorem 659  
C.6 Danilevskii Method for Characteristic Polynomial 660

D Additional Details and Fortification for Chapter 4 664  
D.1 Proofs of Identities of Differential Operators 664  
D.2 Derivation of Formulas in Cylindrical Coordinates 666  
D.3 Derivation of Formulas in Spherical Coordinates 669

E Additional Details and Fortification for Chapter 5 673  
E.1 Line Integrals 673  
E.2 Surface Integrals 678  
E.3 Volume Integrals 684  
E.4 Gauss-Legendre Quadrature 687  
E.5 Proofs of Integral Theorems 691

F Additional Details and Fortification for Chapter 6 700  
F.1 Supplemental Methods for Solving First-Order ODEs 700  
F.2 Singular Solutions 703  
F.3 Finite Series Solution of $dx/dt = Ax + b(t)$ 705  
F.4 Proof for Lemmas and Theorems in Chapter 6 708

G Additional Details and Fortification for Chapter 7 715  
G.1 Differential Equation Solvers in MATLAB 715  
G.2 Derivation of Fourth-Order Runge Kutta Method 718  
G.3 Adams-Bashforth Parameters 722  
G.4 Variable Step Sizes for BDF 723  
G.5 Error Control by Varying Step Size 724  
G.6 Proof of Solution of Difference Equation, Theorem 7.1 730  
G.7 Nonlinear Boundary Value Problems 731  
G.8 Ricatti Equation Method 734

H Additional Details and Fortification for Chapter 8 738  
H.1 Bifurcation Analysis 738

I Additional Details and Fortification for Chapter 9 745  
I.1 Details on Series Solution of Second-Order Systems 745  
I.2 Method of Order Reduction 748  
I.3 Examples of Solution of Regular Singular Points 750  
I.4 Series Solution of Legendre Equations 753  
I.5 Series Solution of Bessel Equations 757  
I.6 Proofs for Lemmas and Theorems in Chapter 9 761

J Additional Details and Fortification for Chapter 10 771  
J.1 Shocks and Rarefaction 771  
J.2 Classification of Second-Order Semilinear Equations: $n > 2$ 781  
J.3 Classification of High-Order Semilinear Equations 784
Contents

K Additional Details and Fortification for Chapter 11 . . . . . . . . . 786
   K.1 d’Alembert Solutions 786
   K.2 Proofs of Lemmas and Theorems in Chapter 11 791

L Additional Details and Fortification for Chapter 12 . . . . . . . 795
   L.1 The Fast Fourier Transform 795
   L.2 Integration of Complex Functions 799
   L.3 Dirichlet Conditions and the Fourier Integral Theorem 819
   L.4 Brief Introduction to Distribution Theory and Delta Distributions 820
   L.5 Tempered Distributions and Fourier Transforms 830
   L.6 Supplemental Lemmas, Theorems, and Proofs 836
   L.7 More Examples of Laplace Transform Solutions 840
   L.8 Proofs of Theorems Used in Distribution Theory 846

M Additional Details and Fortification for Chapter 13 . . . . . . . 851
   M.1 Method of Undetermined Coefficients for Finite Difference Approximation of Mixed Partial Derivative 851
   M.2 Finite Difference Formulas for 3D Cases 852
   M.3 Finite Difference Solutions of Linear Hyperbolic Equations 855
   M.4 Alternating Direction Implicit (ADI) Schemes 863

N Additional Details and Fortification for Chapter 14 . . . . . . . 867
   N.1 Convex Hull Algorithm 867
   N.2 Stabilization via Streamline-Upwind Petrov-Galerkin (SUPG) 870
Preface

This book was written as a textbook on applied mathematics for engineers and scientists, with the expressed goal of merging both analytical and numerical methods more tightly than other textbooks. The role of applied mathematics has continued to grow increasingly important with advancement of science and technology, ranging from modeling and analysis of natural phenomenon to simulation and optimization of man-made systems. With the huge and rapid advances of computing technology, larger and more complex problems can now be tackled and analyzed in a very timely fashion. In several cases, what used to require supercomputers can now be solved using personal computers. Nonetheless, as the technological tools continue to progress, it has become even more imperative that the results can be understood and interpreted clearly and correctly, as well as the need for a deeper knowledge behind the strengths and limitations of the numerical methods used. This means that we cannot forgo the analytical techniques because they continue to provide indispensable insights on the veracity and meaning of the results. The analytical tools continue to be of prime importance for basic understanding for building mathematical models and data analysis. Still, when it comes to solving large and complex problems, numerical methods are needed.

The level of exposition in this book is aimed at graduate students, advanced undergraduate students, and researchers in the engineering and science field. Thus the topics were mostly chosen to continue several topics found in most undergraduate textbooks in applied mathematics. We have focused on advanced concepts and implementation of various mathematical tools to solve the problems that most graduate students are likely to face in their research work and other advanced courses.

The contents of the book can be divided into four main parts: matrix theory, vectors and tensors, ordinary differential equations, and partial differential equations. We begin the book with matrix theory because the tools developed in matrix theory form the crucial foundations used in the rest of the book. The next part centers on the concepts used in vector and tensor theory, including the application of tensor calculus and integral theorems to develop mathematical models of physical systems, often resulting in several differential equations. The last two parts focus on the solution of ordinary and partial differential equations. It can be argued that the primary needs of applied mathematics in engineering and the physical sciences are to obtain models for a system or phenomena in the form of differential equations.
and then to be able to “solve” them to predict and understand the effects of changes in model parameters, boundary conditions, or initial conditions.

Although the methods of applied mathematics are independent of computing platform and programs, we have chosen to use MATLAB as a particular platform under which we investigate the mathematical methods, techniques, and ideas so that the approaches can be tested and the results can be visualized. The supplied MATLAB codes are all included on the book’s website, and the reader can modify the codes for their own use. There are several excellent MATLAB toolboxes supplied by third-party software developers, and they have been optimized for speed, efficiency, and user-friendliness. However, the unintended consequences of user-friendly tools can sometimes render the users to be “button pushers.” We contend that students in applied mathematics still need to discover the mechanism and ideas behind the full-blown programs – at least to apply them to simple test problems and gain some basic understanding of the various approaches. The links to the supplemental MATLAB programs and files can be accessed through the link: www.cambridge.org/Co.

The appendices are collected as chapter fortifications. They include proofs, advanced topics, additional tables, and examples. The reader should be able to access these materials through the web via the link: www.cambridge.org/Co. The index also contains topics that can be found in the appendices, and they are given page numbers that continue the count from the main text.

Several colleagues and students have helped tremendously in the writing of this textbook. Mostly, I want to thank my best friend and wife, Faith Morrison, for the support and encouragement and the sacrifice she’s made so that I could finish this extended and personally significant project. I hope the textbook will contain useful information for the readers, enough for them to share in the continued exploration of the methods and applications of mathematics to further improve the understanding and conditions of our world.

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