Portfolio Theory and Risk Management

With its emphasis on examples, exercises and calculations, this book suits advanced undergraduates as well as postgraduates and practitioners. It provides a clear treatment of the scope and limitations of mean-variance portfolio theory and introduces popular modern risk measures. Proofs are given in detail, assuming only modest mathematical background, but with attention to clarity and rigour. The discussion of VaR and its more robust generalizations, such as AVaR, brings recent developments in risk measures within range of some undergraduate courses and includes a novel discussion of reducing VaR and AVaR by means of hedging techniques.

A moderate pace, careful motivation and more than 70 exercises give students confidence in handling risk assessments in modern finance. Solutions and additional materials for instructors are available at www.cambridge.org/9781107003675.

MACIEJ J. CAPIŃSKI is an Associate Professor in the Faculty of Applied Mathematics at AGH University of Science and Technology in Kraków, Poland. His interests include mathematical finance, financial modelling, computer-assisted proofs in dynamical systems and celestial mechanics. He has authored 10 research publications, one book, and supervised over 30 MSc dissertations, mostly in mathematical finance.

EKKEHARD KOPP is Emeritus Professor of Mathematics at the University of Hull, where he taught courses at all levels in analysis, measure and probability, stochastic processes and mathematical finance between 1970 and 2007. His editorial experience includes service as founding member of the Springer Finance series (1998–2008) and the Cambridge University Press AIMS Library Series. He has taught in the UK, Canada and South Africa and he has authored more than 50 research publications and five books.
Mastering Mathematical Finance

Mastering Mathematical Finance is a series of short books that cover all core topics and the most common electives offered in Master’s programmes in mathematical or quantitative finance. The books are closely coordinated and largely self-contained, and can be used efficiently in combination but also individually.

The MMF books start financially from scratch and mathematically assume only undergraduate calculus, linear algebra and elementary probability theory. The necessary mathematics is developed rigorously, with emphasis on a natural development of mathematical ideas and financial intuition, and the readers quickly see real-life financial applications, both for motivation and as the ultimate end for the theory. All books are written for both teaching and self-study, with worked examples, exercises and solutions.


[PF] *Probability for Finance*, Ekkehard Kopp, Jan Malczak, Tomasz Zastawniak

[SCF] *Stochastic Calculus for Finance*, Marek Capiński, Ekkehard Kopp, Janusz Traple


[NMFC] *Numerical Methods in Finance with C++*, Maciej J. Capiński, Tomasz Zastawniak

[SIR] *Stochastic Interest Rates*, Daragh McInerney, Tomasz Zastawniak

[CR] *Credit Risk*, Marek Capiński, Tomasz Zastawniak

[FE] *Financial Econometrics*, Marek Capiński

[SCAF] *Stochastic Control Applied to Finance*, Szymon Peszat, Tomasz Zastawniak

**Series editors** Marek Capiński, AGH University of Science and Technology, Kraków; Ekkehard Kopp, University of Hull; Tomasz Zastawniak, University of York
Portfolio Theory and Risk Management

MACIEJ J. CAPIŃSKI
AGH University of Science and Technology, Kraków, Poland

EKKEHARD KOPP
University of Hull, Hull, UK
To Anna, Emily, Staś, Weronika and Helenka
Contents

Preface ix

1 Risk and return 1
   1.1 Expected return 2
   1.2 Variance as a risk measure 5
   1.3 Semi-variance 9

2 Portfolios consisting of two assets 11
   2.1 Return 12
   2.2 Attainable set 15
   2.3 Special cases 20
   2.4 Minimum variance portfolio 23
   2.5 Adding a risk-free security 25
   2.6 Indifference curves 28
   2.7 Proofs 31

3 Lagrange multipliers 35
   3.1 Motivating examples 35
   3.2 Constrained extrema 40
   3.3 Proofs 44

4 Portfolios of multiple assets 48
   4.1 Risk and return 48
   4.2 Three risky securities 52
   4.3 Minimum variance portfolio 54
   4.4 Minimum variance line 57
   4.5 Market portfolio 62

5 The Capital Asset Pricing Model 67
   5.1 Derivation of CAPM 68
   5.2 Security market line 71
   5.3 Characteristic line 73

6 Utility functions 76
   6.1 Basic notions and axioms 76
   6.2 Utility maximisation 80
   6.3 Utilities and CAPM 92
   6.4 Risk aversion 95
## Contents

7 Value at Risk ........................................ 98  
  7.1 Quantiles ........................................ 99  
  7.2 Measuring downside risk ...................... 102  
  7.3 Computing VaR: examples .................... 104  
  7.4 VaR in the Black–Scholes model .......... 109  
  7.5 Proofs ........................................ 120  

8 Coherent measures of risk ....................... 124  
  8.1 Average Value at Risk ........................ 125  
  8.2 Quantiles and representations of AVaR .... 127  
  8.3 AVaR in the Black–Scholes model .......... 136  
  8.4 Coherence ..................................... 146  
  8.5 Proofs ........................................ 154  

Index .................................................... 159
Preface

In this fifth volume of the series ‘Mastering Mathematical Finance’ we present a self-contained rigorous account of mean-variance portfolio theory, as well as a simple introduction to utility functions and modern risk measures.

Portfolio theory, exploring the optimal allocation of wealth among different assets in an investment portfolio, based on the twin objectives of maximising return while minimising risk, owes its mathematical formulation to the work of Harry Markowitz\(^1\) in 1952; for which he was awarded the Nobel Prize in Economics in 1990. Mean-variance analysis has held sway for more than half a century, and forms part of the core curriculum in financial economics and business studies. In these settings mathematical rigour may suffer at times, and our aim is to provide a carefully motivated treatment of the mathematical background and content of the theory, assuming only basic calculus and linear algebra as prerequisites.

Chapter 1 provides a brief review of the key concepts of return and risk, while noting some defects of variance as a risk measure. Considering a portfolio with only two risky assets, we show in Chapter 2 how the minimum variance portfolio, minimum variance line, market portfolio and capital market line may be found by elementary calculus methods. Chapter 3 contains a careful account of the method of Lagrange multipliers, including a discussion of sufficient conditions for extrema in the special case of quadratic forms. These techniques are applied in Chapter 4 to generalise the formulae obtained for two-asset portfolios to the general case.

The derivation of the Capital Asset Pricing Model (CAPM) follows in Chapter 5, including two proofs of the CAPM formula, based, respectively, on the underlying geometry (to elucidate the role of beta) and linear algebra (leading to the security market line), and introducing performance measures such as the Jensen index and Sharpe ratio. The security characteristic line is shown to aid the least-squares estimation of beta using historical portfolio returns and the market portfolio.

Chapter 6 contains a brief introduction to utility theory. To keep matters simple we restrict to finite sample spaces to discuss preference relations.

Preface

We consider examples of von Neumann–Morgenstern utility functions, link utility maximisation with the No Arbitrage Principle and explain the key role of state price vectors. Finally, we explore the link between utility maximisation and the CAPM and illustrate the role of the certainty equivalent for the risk averse investor.

In the final two chapters the emphasis shifts from variance to measures of downside risk. Chapter 7 contains an account of Value at Risk (VaR), which remains popular in practice despite its well-documented shortcomings. Following a careful look at quantiles and the algebraic properties of VaR, our emphasis is on computing VaR, especially for assets within the Black–Scholes framework. A novel feature is an account of VaR-optimal hedging with put options, which is shown to reduce to a linear programming problem if the parameters are chosen with care.

In Chapter 8 we examine how the defects of VaR can be addressed using coherent risk measures. The principal example discussed is Average Value at Risk (AVaR), which is described in detail, including a careful proof of sub-additivity. AVaR is placed in the context of coherent risk measures, and generalised to yield spectral risk measures. The analysis of hedging with put options in the Black–Scholes setting is revisited, with AVaR in place of VaR, and the outcomes are compared in examples.

Throughout this volume the emphasis is on examples, applications and computations. The underlying theory is presented rigorously, but as simply as possible. Proofs are given in detail, with the more demanding ones left to the end of each chapter to avoid disrupting the flow of ideas. Applications presented in the final chapters make use of background material from the earlier volumes [PF] and [BSM] in the current series. The exercises form an integral part of the volume, and range from simple verification to more challenging problems. Solutions and additional material can be found at www.cambridge.org/9781107003675, which will be updated regularly.