Sampling Theory

Beyond Bandlimited Systems

Covering the fundamental mathematical underpinnings together with key principles and applications, this book provides a comprehensive guide to the theory and practice of sampling from an engineering perspective. Beginning with traditional ideas such as uniform sampling in shift-invariant spaces and working through to the more recent fields of compressed sensing and sub-Nyquist sampling, the key concepts are addressed in a unified and coherent way. Emphasis is given to applications in signal processing and communications, as well as hardware considerations, throughout.

The book is divided into three main sections: first is a comprehensive review of linear algebra, Fourier analysis, and prominent signal classes figuring in the context of sampling, followed by coverage of sampling with subspace or smoothness priors, including nonlinear sampling and sample rate conversion. Finally, sampling over union of subspaces is discussed, including a detailed introduction to the field of compressed sensing and the theory and applications of sub-Nyquist sampling.

With 200 worked examples and over 250 end-of-chapter problems, this is an ideal course textbook for senior undergraduate and graduate students. It is also an invaluable reference or self-study guide for engineers and students across industry and academia.

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To my parents

To Shalomi, Yonatan, Moriah, Tal, Noa and Roei

The beginning of wisdom is to acquire wisdom;
And with all your means acquire understanding.
Proverbs 4:7

Contents

Preface

Digital signal processing (DSP) is one of the most prominent areas in engineering, including subfields such as speech and image processing, statistical data processing, spectral estimation, biomedical applications, and many others. As the name suggests, the goal is to perform various signal processing tasks (e.g., filtering, amplification, and more) in the digital domain where design, verification, and implementation are considerably simplified compared with analog signal processing. DSP is the basis of many areas of technology, and is one of the most powerful technologies that have shaped science and engineering in the past century.

In order to represent and process analog signals on a computer the signals must be sampled with an analog-to-digital converter (ADC) which converts the signal to a sequence of numbers. After processing, the samples are converted back to the analog domain via a digital-to-analog converter (DAC). Consequently, the theory and practice of sampling are at the heart of DSP. Evidently, any technology advances in ADCs and DACs have a huge impact on a vast array of applications.

The goal of this book is to provide a comprehensive treatment of the theory and practice of sampling from an engineering perspective. Although there are many excellent mathematical textbooks on signal expansions and harmonic analysis, our aim is to present an up-to-date engineering textbook on the topic by combining the fundamental mathematical underpinnings of sampling with practical engineering applications and principles. A large part of the book is also devoted to the more recent fields of compressed sensing and sub-Nyquist sampling which are not covered in standard linear algebra or harmonic analysis books. Throughout, we focus on various applications in signal processing and communications. We assume that the reader is familiar with basic signals processing concepts such as filtering and convolution. The intended audience is a senior undergraduate or first-year graduate level class; however, some background in digital signal processing and Fourier analysis should be enough to follow the material. Required background needed in linear algebra is covered in the text. The book can also be used as a reference for engineers, students working in related areas, and researchers from industry and academia. We also believe that the book is suited for self-study as it is largely self-contained.

Sampling theory is a broad and deep subject, and a vivid area of research, with roots going back over a century. It is therefore impossible to cover all the advances and results in this rich area in a single textbook. The point of this book is not to do justice to the beautiful mathematical theory underlying sampling, but rather to bring forth some of the important engineering concepts in a coherent way. We have chosen to focus primarily on uniform sampling in shift-invariant spaces, and on deterministic signals. The important topics of nonuniform sampling, Gabor and wavelet expansions, errors in sampling due to noise, quantization, implicit sampling, and other approximations are only briefly touched upon. Many of these subjects are covered in other textbooks focused on these specific topics, or in the many references provided at the end of the book.

Organization of the book

The book can be broadly divided into three sections:

- Introductory material including motivation, review of linear algebra and Fourier analysis, and survey of signal classes (Chapters 1–5);
- Sampling with subspace or smoothness priors, including nonlinear sampling and sampling rate conversion (Chapters 6–9);
- Sampling over union of subspaces, including a detailed introduction to the field of compressed sensing and the theory and applications of sub-Nyquist sampling (Chapters 10–15).

We begin in Chapter 1 with a brief introduction to the topic of sampling in general, its importance, and the necessity to move beyond the traditional Shannon–Nyquist theorem. Chapter 2 contains a comprehensive review of the linear algebra background needed in order to develop the mathematical notions underlying sampling theory. In it we have attempted to summarize the main mathematical machinery required for the rest of the book. A fundamental understanding of linear algebra is key to developing sampling theories, and therefore this chapter is quite extensive. Chapter 3 summarizes important notions regarding linear time-invariant systems and Fourier transforms. We review both the continuous-time and discrete-time Fourier transforms, and discuss the relationship between the two in the context of sampled signals. The classes of signals that we will focus on throughout the book are introduced in Chapter 4, along with some of the fundamental mathematical properties associated with such signal sets. In particular, we discuss the celebrated Shannon–Nyquist theorem, and its extension to more general shift-invariant subspaces. We briefly consider Gabor and wavelet expansions, and introduce union of subspaces and smoothness priors. Our primary focus in this book is on signal models involving shift-invariant (SI) spaces. We therefore devote Chapter 5 to studying some of the mathematical properties associated with these spaces. Examples include bandlimited signals, splines, and many classes of digital communication signals.

In Chapter 6 we turn to treat specific sampling theorems. We begin by considering linear sampling with subspace priors. As we show, in many cases perfect recovery of the signal from the given samples is possible, based on the subspace prior, even when the input signal is not bandlimited or the sampling rate is lower than the Nyquist rate. We also treat the case in which constraints are imposed on the recovery process and consider different criteria to recover or approximate the original signal in these settings. In particular, we develop the well-known Papoulis' generalized sampling theorem as a special case of our framework. These ideas are extended in Chapter 7 to include smoothness priors, namely, when all we know about the signal is that it is smooth in some sense. An interesting special case that we treat in this context is super-resolution: obtaining a high-resolution image from several low-resolution images by using ideas of sampling and reconstruction. Nonlinear sampling is considered in Chapter 8, assuming a subspace prior. Surprisingly, we will see that many types of nonlinearities that are encountered in practice can be completely compensated for without having to increase the sampling rate, even though typically nonlinearities lead to an increase in bandwidth. Although sampling theory is focused on recovery of continuous-time signals from their discrete samples, in Chapter 9 we demonstrate that sampling also plays a crucial role in the design of fully discrete-time algorithms in the context of sampling rate conversion. We will discuss several methods for converting between signals or images at different rates. This allows us, in particular, to efficiently vary the size of an image or an audio file.

Chapters 10–15 are devoted to sub-Nyquist sampling and compressed sensing. In Chapter 10 we introduce the union of subspaces (UoS) model, which underlies many sub-Nyquist sampling paradigms. This model allows for nonlinear signal classes which can describe, for example, streams of pulses with unknown delays and amplitudes, multiband signals with unknown carrier frequencies, and more. One of the most well-studied examples of a UoS is that of a vector that is sparse in an appropriate subspace. This model is the basis of the rapidly growing field of compressed sensing, which we review in detail in Chapter 11. This material is based on the chapter "Introduction to compressed sensing," co-authored by M. Davenport, M. Duarte, Y. C. Eldar, and G. Kutyniok, which appears in the book *Compressed Sensing* (Cambridge, 2012). Chapter 12 considers an extension of the basic sparsity model to block sparsity, which can be used to describe more general finite-dimensional unions. This chapter also discusses how to learn the subspaces from subsampled data, when they are not known a priori. Unions of shift-invariant spaces are treated in Chapter 13 along with applications to low-complexity detectors in various settings. The class of multiband signals is considered in Chapter 14. These are signals whose Fourier transform comprises a small number of bands, spread over a wide frequency range. We present a variety of different methods that allow such signals to be sampled at sub-Nyquist rates proportional to the actual band occupancy, even though the carrier frequencies are unknown, and not to the high Nyquist rate associated with the largest frequency. Along with developing the theoretical concepts, we also address practical considerations and demonstrate a hardware realization of a sub-Nyquist sensing board for multiband signals. Chapter 15 is focused on sub-Nyquist sampling of pulse streams which appear in applications such as radar, ultrasound, and multipath channel identification. Example hardware prototypes for problems in radar and ultrasound are also presented.

The appendices cover basic material used in various parts of the book. Specifically, Appendix A summarizes key results related to matrix algebra, and Appendix B reviews basic concepts from probability theory and random processes.

Not all theorems in the book are proven in detail. When proofs are not included, we provide references to where they can be found. Furthermore, in some places, mathematical rigor has been replaced by emphasis on the main ideas.

Matlab implementations and examples

The book contains many worked examples in order to provide deeper understanding and greater intuition for the material, and to illustrate the main points, as well as to explore the behavior of the different methods, and various tradeoffs relevant to the problems at hand. Numerical results are also sometimes used to illustrate points that are not developed rigorously in the text. The numerical experiments have all been programmed in Matlab using standard toolboxes. Numerical examples and computational figures can be reproduced using the m-files available on the author's web page.

At the end of each chapter there is also a list of homework exercises which further expand on and demonstrate the various concepts introduced, and provide an opportunity to practice the material. Some of the exercises are used to derive proofs of theorems that were omitted in the text itself. The order of the exercises follows the presentation of the material in the chapter.

Teaching

This book is intended as a senior year or graduate textbook. It has emerged from teaching "Generalized Sampling Methods" at the Technion – Israel Institute of Technology, and from several tutorials delivered and written on these topics.

Electrical engineering students are often deterred by the vector space formulation of linear algebra used throughout the book. We are accustomed to filtering and convolutions, and manipulation of finite-dimensional matrices. However, much of the beauty of the results in this field comes from the Hilbert space structure. Once these structures are understood, the rest of the results follow naturally and simply. As we will see, proper understanding of these concepts also ultimately leads to simple and efficient hardware. It is therefore very worthwhile to go through the experience of truly comprehending and appreciating linear algebra. Accordingly, the book begins by providing an overview of the essential ingredients in linear algebra needed for the presentation of the material. When teaching this course at the Technion, we dedicate the first few weeks to covering linear algebra basics in depth before delving into sampling theory. In our opinion, beginning with a review of linear algebra is essential. Although all engineering students take basic linear algebra, such courses are typically taught from a matrix-oriented point of view. The more abstract viewpoint advocated here is essential for the chapters to follow, and often new to the students.

The chapter on Fourier analysis can typically be skipped, with only a short reminder of the essential results. In particular, discrete – continuous relations, which are often overlooked but key in the development of sampling results, may be emphasized.

The rest of the book is designed to provide flexibility in how to present the material. The book can be used as a basis for a broad class in sampling theory which covers all the topics in the book – focusing in class on the main results and relying on the book to fill in the details regarding proofs, examples, and applications. On the other hand, one can choose to cover only a subset of the chapters, in greater detail.

As we outlined in the section discussing the book structure, the book is conveniently divided into three sections. The first provides a comprehensive overview of the basic building blocks needed in order to understand and develop subsequent material. These chapters are provided mainly for reference. In a course, most of this material can be skipped, focusing only on the essential concepts which the students in the course may be lacking. As an example, in teaching this course at the Technion, we devote about four classes to linear algebra and shift-invariant spaces; subspace sampling is covered in one class, two classes are dedicated to smoothness priors and interpolation methods, and one class is dedicated to nonlinear sampling. The remaining six weeks of the course focus on compressed sensing and sub-Nyquist sampling, of which about one week is devoted to each of Chapters 10, 11, 13, and 14, and two classes to Chapter 15 and some of its applications.

Alternatively, a semester-length course can focus on the core material in Chapters 5–9, complemented by selected material from Chapters 10–15 as time permits. Most of these chapters can be taught independently of each other.

The book may also be used for a course focused more on the recently growing field of compressed sensing and sub-Nyquist sampling. In this case, the course can begin with a brief introduction to linear algebra and concepts of shift-invariant spaces, and then go through the last unit of the book, i.e. Chapters 10–15, in more detail.

Thanks

Completing this book would not have been possible without the help of many people throughout the multiple stages of the book's evolution. During my years in academia I have been surrounded by good friends and colleagues who have encouraged and supported me. I am very grateful to my colleagues whom I had the pleasure to work with and from whom I learned a great deal about sampling theory and compressed sensing in particular and about research and teaching more generally. I am also indebted to my friends and family, who do not share my passion and interest in engineering and math, and have therefore made sure to provide ample opportunities to be reminded of the many other aspects of life, giving me the energy to continue and the distraction I needed at the many stumbling points during this project.

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current and former graduate students for their contributions to this book through their research results, and for the opportunity to learn from each of them during this process. I apologize for any errors and inconsistencies remaining in the book, and for any omitted subjects which deserved better coverage.

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In recent years we have been working extensively on applications of sampling theory in a wide variety of areas. I have been very fortunate to have brilliant and dedicated colleagues to collaborate with, who are experts in the respective application domains. They have been a tremendous source of inspiration and support and have made research a fun and rewarding experience. Special thanks to Amir Beck, Emmanuel Candes, Israel Cidon, Oren Cohen, Alex Gershman, Andrea Goldsmith, Alex Haimovich, Arye Nehorai, Guillermo Sapiro, Anna Scaglione, Moti Segev, Shlomo Shamai, and Joshua Zeevi. I am very grateful to my excellent hosts during my Sabbatical at Stanford University – Emmanuel Candes at the Statistics department and Andrea Goldsmith at the Electrical Engineering department. My Sabbatical provided many opportunities for working on the book and was full of fun, stimulating, and interesting discussions. Many of my colleagues mentioned above are now personal friends with whom I share more than just our joint passion for research. I would also like to mention my colleagues at the Technion, Gitti Frey, Idit Keidar, Ayellet Tal, and Lihi Zelnik-Manor, who have provided a safety net that helped keep my sanity while trying to balance family life with a demanding career. I am further grateful to all my collaborators throughout the years from whom I have learned a lot about research in general, and signal processing in particular. The Electrical Engineering Department at the Technion has provided an exciting and stimulating environment for both research and teaching during the past 10 years.

In 2013 we established the SAMPL laboratory – Sampling, Acquisition, Modeling and Processing Lab – at the Electrical Engineering Department in the Technion. The sub-Nyquist prototypes presented in this book, as well as many other sub-Nyquist projects, were all developed in the laboratory. I have been extremely fortunate to have the support and expertise of many truly talented engineers. Special thanks to Yoram Or-Chen, Alon Eilam, Rolf Hilgendorf, Alex Reysenson, Idan Shmuel, and Eli Shoshan. The laboratory would not have been established and operative without the support of Peretz

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Special thanks to my parents who inspired me from an early age to follow my ambitions and who have taught me values through a constant living example: to my mother for instilling in me a passion for life, the energy to pursue my goals and for finding solutions to many of life's problems, and to my father for implanting in me the love of knowledge, drive for perfection and excellence, and for his constant sound advice and inspiration. My in-laws have brought me into their family as one of their own and have taken pride in all my accomplishments. I am sincerely grateful to my parents and in-laws as well as to my extended family for their continuing love and support.

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Abbreviations

ADC	Analog-to-digital converter
AM	Amplitude modulation
AWG	Arbitrary waveform generator
AWR	Applied wave research
BCS	Blind compressed sensing
BIBO	Bounded-input, bounded-output
BK-SVD	Block K-SVD
BMP	Block matching pursuit
BOMP	Block orthogonal matching pursuit
BP	Basis pursuit
BPDN	Basis pursuit denoising
C-HiLasso	Collaborative HiLasso
CLS	Constrained least squares
CPI	Coherent processing interval
CPM	Continuous-phase modulation
CR	Cognitive radio
CRB	Cramér–Rao bound
CS	Compressed sensing
CTF	Continuous to finite
CTFT	Continuous-time Fourier transform
DAC	Digital-to-analog converter
DC	Direct-current
DCT	Discrete cosine transform
DFT	Discrete Fourier transform
DL	Dictionary learning
DSP	Digital signal processing
DTFT	Discrete-time Fourier transform
ESPRIT	Estimation of signal parameters by rotational invariance
FIR	Finite-impulse response
FM	Frequency-modulation
FRI	Finite rate of innovation
FUS	Finite union of subspaces
GHz	Gigahertz
IHT	Iterative hard thresholding

IIR	Infinite impulse response
IMV	Infinite measurement vector
ISI	Intersymbol interference
LPF	Low-pass filter
LS	Least squares
LTI	Linear time-invariant
MF	Matched filter
MIMO	Multiple-input, multiple-output
MMSE	Minimum mean-squared error
MMV	Multiple measurement vector
MOD	Method of direction
MP	Matching pursuit
MSE	Mean-squared error
MUD	Multiuser detection
MUSIC	Multiple signal classification
MWC	Modulated wideband converter
NI	National instruments
NSP	Null space property
OBD-BCS	Orthonormal block diagonal BCS
OMP	Orthogonal matching pursuit
PAM	Pulse AM
PRI	Pulse repetition interval
PSD	Power spectrum density
PSF	Point-spread function
PSNR	Peak SNR
QAM	Quadratic amplitude modulation
RDD	Reduced-dimension
RDDF	Reduced-dimension decision-feedback
RD-MUD	Reduced-dimension multiuser detector
RF	Radio frequency
RIP	Restricted isometry property
RKHS	Reproducing kernel Hilbert space
SAC	Sparse agglomerative clustering
SI	Shift-invariant
SIC	Successive interference cancelation
SMI	Shift and modulation invariant
SNR	Signal to noise ratio
SOCP	Second-order cone program
SVD	Singular value decomposition
TLS	Total least squares
ULS	Underspread linear system
UoS	Union of subspaces
WSS	Wide-sense stationary