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Introduction

The second edition of Alfred N. Whitehead and Bertrand Russell's *Principia Mathematica* was published by Cambridge University Press in three volumes between 1925 and 1927. It consists of a reprint of the first edition, which appeared between 1910 and 1913, with the addition of a new Introduction and three Appendices (A, B, and C) written by Russell alone, and a List of Definitions.[1] The new material takes up only 66 pages, yet it proposed radical changes to the system of *Principia Mathematica*, some of which require fundamental rethinking of the nature of logic.

What Russell oddly introduces as the "... most definite improvement resulting from work in mathematical logic during the past fourteen years ... " is the proposal to replace the familiar fundamental logical connectives "or" and "not" with the single "Sheffer stroke", "not-both." This technically trivial change is in fact carried out in a rigorous fashion and does not require any rewriting of the body of *Principia Mathematica* in order to be properly implemented. A second and genuinely fundamental change is the adoption of "extensionality" in the second edition. This requires that all propositional connectives are to be truth-functional, and that co-extensive propositional functions, true of the same arguments, are identified. Russell characterizes this doctrine as the result of two theses, that:

> ... functions of propositions are always truth-functions, and that a function can only occur in a proposition through its values. (*PM*, p.xiv)

What this change amounts to, and how it fits with the details of the various traces of the non-extensional system of the first edition that are unaltered, such as the definition of identity, will be discussed below. The third major change in the second edition is the proposal to abandon the axiom of reducibility in the development of mathematics that is the project of *Principia Mathematica*. The

[1] The new material was all added to the first volume as pages numbered $xiii$ to $xlvi$, at the beginning, following the Preface from the first edition, and the appendices at the end, pages 635 to 666 at the end of the volume. For complete bibliographical information on the editions of *PM*, see Blackwell & Ruja (1994, pp.19–25).

logical types of propositional functions, the subject matter of the higher order logic of *PM*, are distinguished in what came to be called the "ramified" theory of types. Not only is there the distinction between individuals, functions of individuals, and functions of such functions, and so on, which is familiar from Frege's logic, but there is a further division of *orders* of functions required in order to observe the "vicious circle principle" that Russell decided was the key to resolving the paradoxes of both set theory and logic. To use Russell's example, the property of "having all the qualities that make a great general" is represented by a propositional function of a higher order than the lowest order, or "predicative" functions in terms of which it is defined. The axiom of reducibility asserts that there is a predicative function which is nevertheless co-extensive with any given function of a higher order. Russell adopted the axiom of reducibility reluctantly in the first edition of *PM* and it was the target of many criticisms by contemporary logicians almost immediately after 1910. In the second edition, in Appendix B, Russell proposes a proof of the principle of induction in the new system of *Principia Mathematica*, which does not rely on the axiom of reducibility. This proof was criticized by Kurt Gödel in his famous essay "Russell's mathematical logic" (1944), and the project of deriving the principle of induction in certain systems of extensional ramified theory of types without an axiom of reducibility was shown to be impossible. How Russell could have made the elementary mistake in Appendix B that he does, and what was the precise nature of the revised theory of types in which the proof was to proceed, however, have been unresolved puzzles. All three of these significant changes in the second edition have thus, in various ways, been the source of uncertainty about what exactly Russell was intending. The publication of this manuscript material provides some additional clues to resolve these technical questions.

The Bertrand Russell Archives at McMaster University in Canada possess the original manuscripts of all that new material, as well as an additional 53 unused or revised leaves left from a first draft, and 62 leaves of notes in symbolic notation consisting of rejected attempts at the formal results in Appendix B. Only two and one half pages of the manuscript material for the first edition have survived, and no material so close to a final version.[2] The last six chapters of this book contain transcriptions of all of that material using the fonts of the word processing system LaTeX. The first seven chapters include a discussion of the history of the writing of the second edition, and present some additional, previously unpublished material from the Bertrand Russell Archives and the Carnap Archives. Further chapters summarize the notation and content of the technical portions of the first edition that are

[2] See Linsky & Blackwell (2006).

relevant to the new material. The Introduction to the second edition and the three appendices are summarized, with special attention to the improvements that Russell thought should be made to the first edition. A significant technical issue arises from the criticisms made by Kurt Gödel and John Myhill against the argument in Appendix B. The publication of the leaves of notes for Appendix B helps to resolve those technical questions, although some serious mysteries remain for others to resolve with the careful study of this material.

This work has two purposes: specifically, to make this archival material available in print, and more generally, by that means, to restore the reputation of the second edition of *Principia Mathematica* as a serious contribution to logic. Soon after its publication in 1927 the second edition was superseded by other developments in the field of mathematical logic. Although the new material in the second edition, with the exception of Appendix B, is reprinted in the most widely available version of the work, the paperback *Principia Mathematica to ∗56* from 1962, that new material for the second edition is not studied carefully, and is indeed seen as an unfortunately unsuccessful attempt of Russell's to keep up with a subject that had passed him by. Study of the published and archival material, however, shows that the second edition reveals deep issues about the move from the intensional logic of propositional functions in the "ramified theory of types" of the first edition, to the altered theory of types in an extensional logic that Russell saw as an improvement. The failure of his technical proposal about the derivation of the principle of mathematical induction in Appendix B, and its criticism by Gödel, reveal the nature of the move from ramified type theory to set theory formulated in first order, extensional logic that became the new preferred foundation for mathematics. The second edition of *Principia Mathematica* marks the end of logicism as the leading program in the foundations of mathematics, and the rise of the mathematical logic of Gödel and Tarski as its replacement, and so it is important for understanding the history of logicism.

Principia Mathematica is arguably the most important work in the subject of symbolic logic which emerged at the beginning of the twentieth century. In its three volumes there are laid out a cumulative series of definitions and formal proofs by which much of the elementary portions of arithmetic, set theory and the theory of real numbers is deduced in a rigorous fashion. Logic conducted in this way soon became a branch of mathematics, to be called "mathematical logic" and so *Principia Mathematica* suffered the fate of many important works in the history of mathematics. Improvements were made in notation and definitions, and better and more rigorous systems of proof were developed. By the 1930s, Kurt Gödel and others had isolated interest in a fragment of the system, the "first order", extensional quantificational logic which is now the subject of countless textbooks and taught as a part of the undergraduate curriculum in philosophy, and as part of elementary courses in computing science in universities around the world.[3]

Gödel's work realized the notion of formal "meta-logic" that had been first proposed by Hilbert and others, namely the creation of a mathematical theory of sentences and symbolic formulas which allowed the derivation of theorems about the strengths and limitations of symbolic logic. He proved that standard systems of first order, extensional logic were complete, in the sense that every "valid" sentence, in a precisely definable sense, is in fact provable, with a precise formal definition of the notion of proof. He further refined the notion of a mechanically verifiable, rigorous, proof, and so led the way to the isolation of the notion of computable algorithm that became the theory that led to the invention of programmable computers. Using the new notion of a provable sentence, he showed that no first order theory of mathematics could be complete in the sense of being capable of proving all truths expressible in that theory. His model for such a theory is presented in the title of his famous paper, "On formally undecidable propositions of *Principia Mathematica* and related systems I" (Gödel, 1931, p.596). The project of *PM* of reducing mathematics to logic thus seemed impossible. Some other approach to mathematics, beginning again with precise, first order axiomatic formal theories of arithmetic and set theory, seemed to be the proper way to investigate the foundations of mathematics. Even in the classic history of logic, *The Development of Logic* by Kneale & Kneale (1962), this result was still considered to show that the logicist project of *Principia Mathematica* was necessarily a failure.[4]

Also in the 1930s Alfred Tarski developed the foundations of the new field of formal semantics, most classically with his "semantic conception of truth", which enabled a mathematical investigation of the definability of the concept of truth.[5] With the subsequent development of a precise concept of mathematical models, the notion of truth in a model, and so of logical consequence via truth in certain models, and the definition of validity as truth in all models, the way was open for even elementary logic to turn from the construction of proofs to the study of classes of sentences in terms of the models in which they are true. With the notion of a tautology of propositional logic clearly defined, the details of the actual proofs of theorems, which is the subject matter of ∗1 to ∗9 of *Principia Mathematica*, was seen as a merely tedious exercise. The rise of set theory, as the language in which the theory of models and the notion of truth were developed, turned out to provide a much simpler development of the elementary parts of mathematics that had taken three full volumes in *PM*.

[3] See Goldfarb (1979), which traces the rise of extensional first order logic, with its model theoretic semantics.
[4] See the discussion of the view that Whitehead and Russell's logicism was a failure in Linsky & Zalta (2006, p.6).
[5] See Tarski (1936).

As a work in mathematics, *Principia Mathematica* soon became obsolete. Symbolic logic is also a field of philosophy, and so the study of major works from the past has a special role unlike that of the study of the history of mathematics. Certainly there were criticisms of *Principia Mathematica* from the point of view of the philosophy of logic. The intensional nature of the logic of *PM*, when it is seen as based on propositions and propositional functions rather than sentences and predicates, and the potential distinction between co-extensive functions, was alien to the extensional account of logic that grew to supplant *PM*. To W.V. Quine, otherwise a champion of *PM*, this intensionality was seen to be a result of an unfortunate, elementary, confusion of use and mention:[6]

> It is regretable that Frege's own scrupulous observance of this distinction between an expression and its name, between use and mention, was so little heeded by Whitehead, Russell, and their critics. (Quine, 1951a, p.142)

The charge of "use–mention" confusions in *PM* goes back to Frege's first reactions upon reading the work. In a letter to Jourdain of 28 January 1914, Frege had complained that:

> I find it very difficult to read Russell's *Principia*; I stumble over almost every sentence. (Frege, 1980, p.81)

A draft of the letter explains the difficulty, "I never know for sure whether he is speaking of a sign or of its content." (Frege, 1980, p.78). Following the criticisms in Wittgenstein's *Tractatus Logico-Philosophicus*, *Principia Mathematica* took on a role in analytic philosophy of logic as a starting point from which progress was made by first correcting its many errors, errors which in many cases were avoided by Frege earlier. In that way *Principia Mathematica* is often seen as a wrong turn in a progression from Frege's *Grundgesetze der Arithmetik* through the *Tractatus* and on into the philosophy and logic of Carnap, Gödel and Tarski.

For all its technical crudities to our eyes, the two editions of *Principia Mathematica* represent the development of certain important ideas in logic. The notions of type theory, extensionality, truth-functionality, the definability of identity and the primitive notions of set theory, and most importantly, the idea of the reduction of mathematics to logic, all change between the two editions of *PM*. The fundamental idea of logic as a theory of intensional propositional functions, instead of a formal system with a certain class of allowable models as interpretations, was abandoned as backward. Studying the history of *Principia Mathematica* reveals important knowledge about the history and philosophy of logic in the first part of

[6] The back cover of the paperback 1962 edition of *Principia Mathematica to ∗56* includes the quote from Quine: "This is the book that has meant the most to me."

the twentieth century. It is even possible to find in the unsuccessful attempts in the new work a defense of the ideas behind the original ramified theory of types as a viable intensional logic, worthy of study today.

My first task, however, is to motivate interest in the second edition of *PM* in the face of the generally dismal assessment, including that of defenders of the original edition. In his book *The Search for Mathematical Roots: 1870–1940*, in which *Principia Mathematica* is the centerpiece of the story, Ivor Grattan-Guinness dismisses the second edition as "hardly a philosophical advance upon the first." (Grattan-Guinness, 2000, p. 443). Ray Monk (2000), in the second volume of his biography, *Bertrand Russell: The Ghost of Madness: 1921–1970* condemns the work as an inadequate attempt to appreciate Wittgenstein's thought which, at the same time, betrayed Whitehead's co-authorship of the first edition:

> First, Russell had not read most of the recent technical literature on the subject, and had neither the time nor the inclination to master it. Second, a complete acceptance of Wittgenstein's work would require, not just changes to the system of *Principia*, but its complete abandonment. Third, as *Principia Mathematica* was co-written with Whitehead, this new edition would also have to be published under both their names, and Whitehead was deeply unsympathetic to Wittgenstein's work and thus to the general lines on which Russell sought to 'improve' their joint undertaking.
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> In the face of these difficulties, what Russell produced was a piece of work unsatisfactory in almost every respect, one that failed to realise any of its aims, made no significant technical advances to the subject and was disliked by both Wittgenstein and Whitehead. (Monk, 2000, p.44)

Russell himself only devotes part of one sentence in his *Autobiography* to his work on the second edition.[7] Very little has been written about the second edition, and most of it has Monk's critical tone. The evidence presented in what follows, from the unused notes, and from a careful rereading of the published version, suggests that a more charitable assessment is more accurate.

Chapter 2 presents the history of the writing of the new material for the second edition. The work was all Russell's, and Whitehead seems to have been content with both the process and the results. Russell began the work in late 1923, with what was intended to be a single long addition to the original Introduction. That first manuscript, identified here as "Hierarchy of propositions and functions", was then split up into the final configuration of a separate Introduction to the second edition and the three appendices. Frank Ramsey read and made some comments on that first manuscript in early 1924, and then read the

[7] The whole sentence is: "There was a new edition of *Principia Mathematica* in 1925, to which I made various additions; and in 1927 I published *The Analysis of Matter* (*AMa*), which is in some sense a companion volume to *The Analysis of Mind* (*AMi*), begun in prison and published in 1921." (*Auto.*, p.214). There is, however, a longer discussion of the second edition in *My Philosophical Development* (*MPD*, pp.120–23).

proofs of portions of the final version in the fall of that year, but all the rest of the work was Russell's.

Chapter 3 describes the development of Russell's ideas about logic since the first edition, both among his own intimates around Cambridge, and in the growing literature in the field. At Cambridge after his dismissal from Trinity College, Russell had a circle of associates working on logic; Henry Sheffer, Jean Nicod and Dorothy Wrinch all wrote on logic and discussed it with Russell. Beginning in 1911, just as the first edition was in press, Ludwig Wittgenstein arrived in Cambridge and remained close to Russell until his departure for Norway in 1913 and his subsequent return to Austria when the war began. After the war Russell assisted Wittgenstein with arranging the publication of the *Tractatus Logico-Philosophicus*, to which Russell contributed an introduction. The evidence suggests that Russell developed his ideas between the editions of *PM* over the whole period, entering into intense discussions with Wittgenstein, but also others, and then presenting his evolving views in the Philosophy of Logical Atomism lectures (PLA) in 1918, the *Introduction to Mathematical Philosophy*, 1919, and then his introduction to Wittgenstein's *Tractatus* in 1921. The actual composition of the second edition, during 1923 and 1924, found Russell working alone, with Frank Ramsey reporting to Wittgenstein about what he was reading of it.

Chapter 3 is organized around a discussion of the list of "contributions to mathematical logic since the publication of the first edition of *Principia Mathematica*" which concludes the new introduction. An examination of each of these works in turn contradicts Monk's claim that Russell had neither read the technical literature in his subject nor mastered it. While Russell was aware of developments in mathematical logic, they were developing logic in radically different directions, and so he took from that literature just what was relevant to his alterations to the system of the first edition. The picture that emerges is of new ideas and schools of thought, the Intuitionism of Brouwer, the work of the Hilbert Program, Wittgenstein's *Tractatus*, and others, emerging in reaction to *Principia Mathematica*. Russell was aware of, but did not follow in, these other directions.

Chapter 4 provides a review of the symbolism and content of the first edition of *Principia Mathematica*, in order of their presentation by "numbers", indicated with the asterisk, as in "∗14" and "∗20", so that the chapter can be used as a glossary of definitions and theorems for reference in what follows. All of the symbolism and references in the new material for the second edition should be presented in some form earlier in this text, although the reader will want to have a copy of at least the paperback edition *Principia Mathematica to ∗56* ready to hand.

Chapter 5 describes the proposed changes to *PM* that constitute the system of the second edition. These changes are listed in the Introduction to the second edition, which describes the role of the Sheffer stroke as a preferred primitive connective, and the subsequent theorems needed for that revision, proved in Appendix A. The adoption of extensionality, and accompanying abandonment of the axiom of reducibility, are the main alterations to the system of the first edition. A revision of the theory of types in the first edition is suggested by the account of types in the introductory material, and will be the subject of careful interpretation in the texts.

Chapter 6 studies in detail the content of Appendix B, On induction. The appendix consists of a technical proof that even without the axiom of reducibility, a limited form of the principle of mathematical induction can be derived. This proof was found to contain a technical flaw by Gödel (1944), "Russell's mathematical logic". Later John Myhill (1974) followed this up with a proof showing that the project of Appendix B is impossible in principle. The manuscript material presented here allows for a better understanding of these issues. The "mistake" can be traced through the earlier drafts of the appendix, and a sense of the intended project of the appendix can be seen in the manuscript notes "Amended list of propositions." It appears that the published material in Appendix B was not slipshod or casual, as Myhill suggests, but rather the result of intense efforts, although in the end, the details of the proposed alterations to the logic of the first edition are not clear. A puzzle remains for the reader to solve by careful study of the notes published here.

Chapter 7 is a survey of some of the more prominent reviews of the second edition. It begins with an overview of Frank Ramsey (1926), "The foundations of mathematics" which can be viewed as Ramsey's response to the issues in *PM* about the axiom of reducibility, the nature of quantification and propositional functions which came from his study of Wittgestein's *Tractatus*, and the second edition material which he had read the year before. The reviews of the second edition by Alonzo Church, C. I. Lewis, and others, help to place *Principia Mathematica* in the logical scene in the mid 1920s, although they are directed more at the general program of logicism (identified with Frege and Russell) than the technical issues that are the focus in this work.

There is much still to be learned about the history of logic in this period, and about the nature of intensional logic which was lost in the move towards extensionality that dominated from the 1920s until the great development of interest in intensional logic and the explosion of work on modal logic and possible worlds semantics following Saul Kripke's papers in the early 1960s. Perhaps type theory has yet to enjoy quite such a revival of interest. It is now acknowledged that Gödel's incompleteness theorem, with its attention to systems with recursively enumerable axioms, does not show that "full" second order logic is incomplete, when the predicate quantifiers are interpreted as ranging over all sets of objects in the domain. It will then also not apply to the intensional theory of types of *Principia Mathematica*. The contemporary logical scene may be ready for an investigation of the ramified theory of types as an intensional logic.

1.1 The manuscripts

The first manuscript transcribed below is the list of the principal definitions and theorems in *Principia Mathematica* that Russell wrote out by hand for Rudolf Carnap in 1922. It is published here with the permission of the Archive of Scientific Philosophy at the University of Pittsburgh, which holds Carnap's papers.

The other manuscript material, transcribed beginning after Chapter 8, came to the Bertrand Russell Archives as part of the original purchase of Russell's papers by McMaster University in 1968. Some smaller items, including Russell's personal copy of the first edition of *Principia Mathematica* with some important material relevant to possible revisions stuck between the pages, arrived with the so-called "Second Archive" after Russell's death in 1970.[8] The manuscripts here called "Hierarchy of propositions and functions" and "Amended list of propositions" were catalogued under those names, and kept in files with the manuscript of the introduction and appendices. The *Collected Papers of Bertrand Russell* series, being conducted by the Bertrand Russell Research Centre at McMaster University, is dedicated to publishing papers and some additional manuscript materials, but the plan was that the material relevant to Russell's books would be used in critical editions produced by their publishers. While a scholarly critical edition of the whole of *Principia Mathematica* would be a massive editorial project, this volume may be considered a first attempt at such a presentation of the second edition of *Principia Mathematica*, and so an opportunity to publish further holdings of the Archives.

Russell's manuscripts were clearly his final versions, sent to Cambridge University Press for typesetting, and returned to him for proofreading, and thus surviving among his papers. The few differences between these versions and the printed edition presumably occurred at that proof stage, and are indicated in the transcriptions. The manuscript leaves are written in an easily readable handwriting, as they were intended for the use of typesetters. Those typesetters were put to work on four or five leaf sections of manuscripts, and the individual names of some of them are written in the upper left hand corners by someone else, presumably a chief printer who delegated the work. Russell numbered the manuscript leaves and this numbering can be used to reconstruct the original "Hierarchy of propositions and functions" manuscript. Russell moved material, simply striking out old numbering and adding the new.

The "Amended list of propositions" manuscript is very different. It consists of single leaves of notes in symbolic notation, with no numbering, and it is named after the heading of the first leaf of notes, which is almost the only writing in the whole

[8] See Blackwell & Spadoni (1992).

manuscript not in *Principia* symbolic notation. While this material was identified as connected with *PM* and kept with the manuscripts, its nature as repeated attempts at finding proofs for lemmas in Appendix B has not been previously identified in print.

The manuscripts are transcribed here in such a way that the content of each original can be reconstructed. They are identified with Russell's numbering ("foliation") in italics in the upper right hand corner, with a break between the content of leaves. For the introduction and appendices, the page number in the printed edition is inserted into the text where the new page begins. Russell struck out some material with a single line (reproduced here in the text as ~~deletions~~) and sometimes added some material above the line, and that is indicated as well. These insertions, usually above the line with a caret to indicate their position, are inserted in the line with '⌊' and '⌋' around the inserted material, as follows: ⌊insertions⌋.The editorial conventions are explained at the beginning of Chapter 12. For the introduction and appendices the "house style" of capitalization, italics, and spacing is imposed on the manuscript so that the reader can more easily identify the changes between the final manuscript and the published version. These corrections were likely made by Russell in the proof stage. The unpublished manuscripts "Hierarchy of propositions and functions" and "Amended list of propositions" are reproduced literally so that the reader can see the layout of text and symbols on the page.

1.2 Acknowledgements

Kenneth Blackwell and Nicholas Griffin have guided me throughout these efforts. This work is, however, not an activity of the Bertrand Russell Research Centre, or McMaster, and Blackwell, the Honorary Archivist, and Griffin, Director of the BRRC, are not responsible for lapses from the high scholarly standards and absolute accuracy of all of their work. Blackwell has nevertheless assisted me at every stage of my work. I came to the Archives as a fledgling Russell scholar for the first time in October 2003, asking if there was any unpublished material "in symbols" that I might see. Ken and Nicholas Griffin together presented me with the box of material on the second edition. During the summer of 2010 Blackwell provided suggestions for the editing of the manuscript material and Griffin read the final manuscript correcting numerous errors. Before I even knew of the manuscripts, between January and June of 1997, I had the opportunity to study Appendix B with Allen Hazen at the University of Melbourne. Together we discussed Gregory Landini's then recently published paper (1996), and worked through the appendix, losing and then finding again the "mistake" in line 3 of the proof of ∗89·16. Hazen, Landini and I had occasion to discuss Appendix B further at the conference "One