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Marek Capiński , Tomasz Zastawniak
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Structural models

Consider a company launched at time 0, when some assets are purchased for $V(0)$. Funding comes from two sources. Shareholders contribute $E(0)$, referred to as **equity**. The remaining amount $D(0) = V(0) - E(0)$, called **debt**, is either borrowed from a bank or raised by selling bonds issued by the company.

We consider this company over a time interval from 0 to T, during which the assets are put to work in order to generate some funds, which are then split between the two groups of investors at time T. The debt is first repaid with interest to the debt holders, who have priority over the equity holders. Any remaining amount goes to the equity holders.

The simplest way to raise money to make these payments is to sell the assets of the company. We begin our analysis with this case, by making the necessary assumption that the assets are tradeable.

1.1 Traded assets

We assume that there is a liquid market for the assets, and $V(t)$ for $t \in [0, T]$ represents their market value. We also assume that the assets generate no additional cash flows. A practical example of such assets would be a portfolio of traded stocks that pay no dividends, the company being an investment fund.

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Payoffs

Suppose that the company has to clear the debt at time T, and that there are no intermediate cash flows to the debt holders. The interest rate applying to the loan will be quoted by the bank or implied by the bond price. We denote this **loan rate** by k_D with continuous compounding, and by K_D with annual compounding. The amount due at time T is

$$F = D(0)e^{k_D T} = D(0)(1 + K_D)^T.$$

(Throughout this volume we take one year as the unit of time.) One of the goals here is to find the loan rate that reflects the risk for the debt holders.

At time T we sell the assets and close down the business, at least hypothetically, to analyse the company's financial position at that time. It is possible that the amount obtained by selling the assets is insufficient to settle the debt, that is, $V(T) < F$. In this respect, we make an important assumption concerning the legal status of the company: it has **limited liability**. This means that losses cannot exceed the initial equity value $E(0)$. If $V(T) < F$, the equity holders do not have to cover the loss from their personal funds. The company is declared bankrupt, and the equity holders walk away having lost their initial investment. If $V(T) \geq F$, the loan can be paid back with interest, and the equity holders keep the balance. The final value of equity is therefore a random amount equal to the payoff of a call option,

$$E(T) = \max\{V(T) - F, 0\},$$

with the value of the assets as the underlying security and the debt repayment amount F as the strike price.

Remark 1.1
A call is an option to buy the underlying asset for a prescribed price, which sounds paradoxical here. However, it is consistent with the general practice that loans are secured on some assets. The borrower's ownership rights in the assets are restricted until the loan is repaid. The full rights (for instance, to sell the asset) are in a sense bought back when the loan is settled.

If $V(T) \geq F$, the debt will be paid in full, but in the case of bankruptcy, which will be declared if $V(T) < F$, the debt holders are going to take over the assets and sell them for $V(T)$. This is straightforward if the assets are tradeable. The amount received at time T will be

$$D(T) = \min\{F, V(T)\} = F - \max\{F - V(T), 0\}.$$

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We immediately recognise the put payoff as one of the components,

$$P(T) = \max\{F - V(T), 0\}.$$

It reflects the limited liability feature.

Observe that

$$D(T) + E(T) = \min\{F, V(T)\} + \max\{V(T) - F, 0\} = V(T).$$

This is similar to the equality $V(0) = D(0) + E(0)$, which holds at time 0, and is called the **balance sheet equation**. It illustrates one of the basic rules of corporate finance: the assets are equal to the liabilities (debt plus equity).

If the payoff of the put option is identically zero (which is possible, for example in the binomial model when the strike price is low enough), then $D(T) = F$ and the debt position is risk free. In this case, we should have $k_D = r$, the continuously compounded risk-free rate. If it is possible that the debt holders recover less than F, a higher rate k_D will be applied to compensate for the risk.

These remarks motivate the following proposition, which does not depend on any particular model for the asset value process. Let $P(0)$ denote the time 0 price of the put option.

Proposition 1.2
If $P(0) > 0$, then $k_D > r$.

Proof Recall the put-call parity relationship expressed in terms of a call, put, stock, and a general strike price K:

$$S(0) = C(0) - P(0) + Ke^{-rT}.$$

In our case it becomes

$$V(0) = E(0) - P(0) + Fe^{-rT}.$$

This implies that

$$D(0) = Fe^{-rT} - P(0).$$

In other words,

$$F = [D(0) + P(0)]\, e^{rT}. \tag{1.1}$$

Recall that $F = D(0)e^{k_D T}$, so

$$D(0)e^{k_D T} = [D(0) + P(0)]\, e^{rT},$$

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which yields

$$k_D = r + \frac{1}{T}\ln\left(1 + \frac{P(0)}{D(0)}\right)$$

and implies that $k_D > r$ as claimed since the second term on the right-hand side is positive. □

As we can see, the loan rate k_D is typically higher than the risk-free rate r. This is consistent with intuition since the loan is not free of risk as the full amount F is paid only in some circumstances.

Definition 1.3
The difference $s = k_D - r$ is called the **credit spread**.

This quantity represents the additional return demanded by the debt holders to compensate for their exposure to default risk.

The positivity of the credit spread is all that can be discovered without specifying a model for asset values. Such a model is needed to have a method of computing option prices, and in turn solving the pivotal problem of setting the level of F, hence k_D, for a given debt and equity values $E(0)$ and $D(0)$, which determine the financial structure of the company.

It is often convenient to describe the financial structure of the company in terms of ratios rather than the actual debt and equity values.

Definition 1.4
The **debt** and **equity ratios** are defined as

$$w_D = \frac{D(0)}{V(0)}, \quad w_E = \frac{E(0)}{V(0)}.$$

Because $V(0) = D(0) + E(0)$, these ratios satisfy $w_E + w_D = 1$.

We have seen that equity can be regarded as a call option with strike F. The time 0 price of this call option, which we now denote by $C(F)$, satisfies $C(F) = E(0)$. This equation can be solved for F, and we illustrate this with the simplest model.

Binomial model

Consider the single-step binomial model and suppose that $V(T)$ takes only two values $V(0)(1 + U)$ and $V(0)(1 + D)$ determined by the returns $-1 < D < R < U$, where R is the risk-free return, that is, $1 + R = e^{rT}$. The non-trivial range of strike prices is

$$V(0)(1 + D) < F < V(0)(1 + U),$$

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and then

$$E(0) = C(F) = \frac{1}{1+R} q(V(0)(1+U) - F),$$

where

$$q = \frac{R-D}{U-D}$$

is the risk-neutral probability (see [DMFM]). The formula is justified by the fact that the payoff of the call option can be replicated by means of $V(T)$ and the risk-free asset, both assumed tradeable. This gives us the corresponding range of initial equity values

$$0 < E(0) < \frac{1}{1+R} qV(0)(U-D).$$

The equation for F can be solved to get

$$F = V(0)(1+U) - \frac{1}{q} E(0)(1+R),$$

and then

$$e^{k_D T} = (1+K_D)^T = \frac{F}{D(0)}.$$

The investors are interested in real-life probabilities to evaluate their prospects, and these should be used to find the expected returns and standard deviations of returns for equity and debt. The computations are straightforward, and we simply consider a numerical example.

Example 1.5
Let $V(0) = 100$, $T = 1$, $R = 20\%$, $U = 40\%$, and $D = -40\%$, hence $q = 0.75$. With $E(0) = 40$ we find $F = 76$ and $K_D = 26.67\%$. Assuming the real-life probability of the up movement to be $p = 0.9$, we get the expected return on assets to be $\mu_V = 32\%$, the expected return on equity $\mu_E = 44\%$, and on debt $\mu_D = 24\%$. The last figure is important for the debt holders as it will be earned on average if the loan rate K_D is quoted for all similar customers. The standard deviation of the return on debt is $\sigma_D = 8\%$.

If equity is reduced to $E(0) = 20$, we have $F = 108$, $K_D = 35\%$, and the expected return on debt under the real-life probability grows to 29% while the standard deviation of the return on debt becomes 18%.

In the extreme (and unrealistic) case of $E(0) = 50$ we have $K_D = R$ and the expected return is the same, the risk being zero (the debt payoff

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is F in each scenario). In the other extreme case of $E(0) = 0$ the company is owned entirely by the debt holders, the parameters for debt coinciding with those for the assets.

Since the payoff for equity is an affine function of $V(T)$, the expected return on equity is the same for each level of financing, $\mu_E = 44\%$. The model is not sophisticated enough to see anything interesting here.

Remark 1.6

In the single-step binomial model the balance between the expected return μ_H and standard deviation σ_H of return of any derivative security H (in particular $H = V$, E, or D) is captured by the fact that the market price of risk $\frac{\mu_H - R}{\sigma_H}$ is the same for each H; see [DMFM].

For two steps the situation becomes more interesting. There are three possible values of $V(T)$ and larger scope for non-trivial cases. The range for the strike price is

$$V(0)(1 + D)^2 < F < V(0)(1 + U)^2.$$

The pricing formula and, in particular, the equation for F become more complicated. Once again, we just analyse a numerical example.

Example 1.7

Using the data from Example 1.5, we perform computations for two cases, $E(0) = 40$ and $E(0) = 60$. We find the respective values of F to be 93.60 and 59.04, the expected two-period returns on equity 107.36% and 92.38%, and the corresponding standard deviations 100.43% and 74.20%. The corresponding expected returns on debt are 52.16% and 47.02% (as compared with the risk-free return of 44%), with standard deviations 11.11% and 5.73%, respectively.

Exercise 1.1 Derive an explicit general formula for F in the two-step binomial model, and compute the expected return and standard deviation of the return for equity and debt as in Example 1.7 in the case of 50% financing by equity.

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Rather than considering the n-step binomial model and using the Cox–Ross–Rubinstein (CRR) formula for the call price (see [DMFM]), which is similar to the Black–Scholes formula, we proceed directly to the latter as it is important and in fact easier to handle.

1.2 Merton model

Suppose that the assets of a company are tradeable and follow the Black–Scholes model, i.e. satisfy the stochastic differential equation

$$dV(t) = \mu V(t)dt + \sigma V(t)dW_P(t),$$

where $W_P(t)$ is a Wiener process under the real-life probability P. Girsanov's theorem (see [BSM]) makes it possible to change to the risk-neutral probability Q and write the stochastic differential equation as

$$dV(t) = rV(t)dt + \sigma V(t)dW_Q(t),$$

where W_Q is a Wiener process under Q.

Let us put

$$C(V(0), \sigma, r, T, F) = V(0)N(d_+) - e^{-rT} FN(d_-),$$

where

$$d_+ = \frac{\ln \frac{V(0)}{F} + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \qquad d_- = \frac{\ln \frac{V(0)}{F} + (r - \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}}, \tag{1.2}$$

and where

$$N(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}y^2} dy$$

is the standard normal cumulative distribution function. We recognise the expression defining $C(V(0), \sigma, r, T, F)$ as the Black–Scholes call pricing formula; see [BSM].

We have $E(0) = C(V(0), \sigma, r, T, F)$ since equity is a call option with strike F. This can be written as

$$E(0) = V(0)N(d_+) - e^{-rT} FN(d_-).$$

The equation needs to be solved for F numerically, with $V(0)$, σ, r, and T fixed. (When solving the equation, remember that d_+ and d_- also depend on F.) The formula for the initial value of debt reads

$$D(0) = V(0)N(-d_+) + e^{-rT} FN(d_-).$$

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This follows from the balance sheet equation $V(0) = E(0) + D(0)$ and the symmetry $1 - N(x) = N(-x)$ of the standard normal distribution. Together, these are the ingredients of **Merton's model** of credit risk.

Example 1.8

Let $V(0) = 100$ and consider 50% financing by equity. Assume the risk-free rate $r = 5\%$ and volatility $\sigma = 30\%$, and take $T = 1$. We can solve the equation

$$C(V(0), \sigma, r, T, F) = 50$$

to find $F = 52.6432$, and then compute the loan rate

$$k_D = \frac{1}{T} \ln \frac{F}{D(0)} = 5.1515\%,$$
$$K_D = e^{k_D} - 1 = 5.2865\%.$$

Exercise 1.2 Within the setup of Example 1.8 consider an investment in stock with volatility higher than 30%. What does your intuition say about the impact of this on k_D? Analyse the monotonicity of k_D as a function of σ. Perform numerical computations for $\sigma = 35\%$.

Expected returns

It is interesting to find the expected returns on equity and debt between the time instants 0 and T under the real-life probability P and analyse their dependence on the financial structure. To this end we need to compute the expectation $\mathbb{E}_P(E(T))$ under the real-life probability. The Black–Scholes formula gives a similar expectation but under the risk-neutral probability,

$$\mathbb{E}_Q(E(T)) = \mathbb{E}_Q((V(T) - F)^+) = e^{rT} C(V(0), \sigma, r, T, F),$$

where

$$V(T) = V(0) \exp\left(\left(r - \frac{1}{2}\sigma^2\right) T + \sigma W_Q(T)\right).$$

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This formula is valid for every $r > 0$, in particular for $r = \mu$. On the other hand, we also have

$$V(T) = V(0)\exp\left(\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma W_P(T)\right).$$

Because $W_Q(T)$ has the same probability distribution under the risk-neutral probability Q as $W_P(T)$ under the real-life probability P (namely the normal distribution $N(0,T)$), it follows that

$$\mathbb{E}_P(E(T)) = \mathbb{E}_P((V(T) - F)^+) = e^{\mu T} C(V(0), \sigma, \mu, T, F).$$

This enables us to compute the expected return on equity under the real-life probability,

$$\mu_E = \frac{\mathbb{E}_P(E(T)) - E(0)}{E(0)}.$$

To find the expected return on debt μ_D we can use the relationship

$$\mu_V = w_E \mu_E + w_D \mu_D$$

from portfolio theory (see [PTRM]), with $\mu_V = e^{\mu T} - 1$ since $\mathbb{E}_P(V(T)) = V(0)e^{\mu T}$.

Example 1.9
For the data from Example 1.8 and $\mu = 10\%$ we get $\mu_V = 10.52\%$, $\mu_E = 15.85\%$, and $\mu_D = 5.19\%$.

Exercise 1.3 Using the data in Example 1.8 and $\mu = 10\%$, compute F and μ_E, μ_D for a company with 40% financing by equity, and also for one with 60% financing by equity.

Example 1.10
For the data in Example 1.8 and $\mu = 10\%$, in Figure 1.1 we plot the graphs of the expected returns μ_E and μ_D as functions of the equity ratio w_E. For comparison, we include the expected return $\mu_V = e^{\mu T} - 1$ on the assets, independent of financing, thus a horizontal line.

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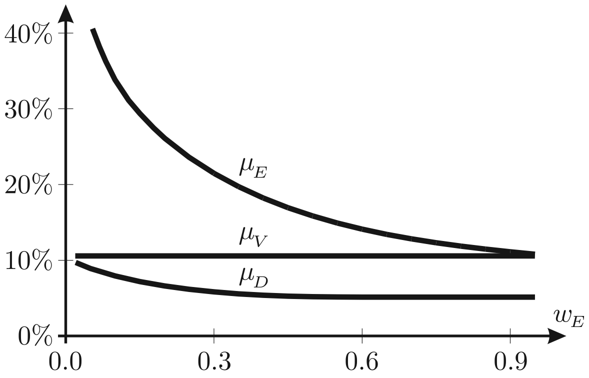

Figure 1.1 Expected returns μ_V, μ_E, μ_D as functions of equity ratio w_E.

High level of debt is profitable for equity holders, and it also appears attractive to debt holders. However, in real life the amount of debt recovered following bankruptcy will be reduced by legal costs. In addition, rapid liquidation of a large number of assets may reduce the prices. These factors affect the debt payoff and hence the expected return on debt μ_D, computed above assuming full recovery.

Partial recovery

We are going to discuss the case when the market value of the company's assets cannot be fully recovered due to the cost of bankruptcy procedures. It is not possible to use the relationship $\mu_V = w_D\mu_D + w_E\mu_E$ from portfolio theory because additional participants emerge in the case of bankruptcy, such as bailiffs or legal services providers.

Suppose that the amount recovered by debt holders is proportional to the value of the assets. When default occurs, that is, when $V(T) < F$, bankruptcy procedures are initiated, the assets are sold, and the debt holders receive $\alpha V(T)$, where $\alpha \in [0, 1]$ is a constant **recovery rate**. Otherwise, when $V(T) \geq F$, the company remains solvent and able to settle the debt in full. As a result, the debt payoff becomes

$$D(T) = F\mathbf{1}_{\{V(T)\geq F\}} + \alpha V(T)\mathbf{1}_{\{V(T)<F\}}.$$

When $\alpha < 1$ the debt holders' payoff $\alpha V(T)$ in the case of bankruptcy is reduced as compared to the full recovery payoff $V(T)$ in the case with $\alpha = 1$. To be compensated for this reduction, the debt holders will demand

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