Stochastic Calculus for Finance

This book focuses specifically on the key results in stochastic processes that have become essential for finance practitioners to understand. The authors study the Wiener process and Itô integrals in some detail, with a focus on results needed for the Black–Scholes option pricing model. After developing the required martingale properties of this process, the construction of the integral and the Itô formula (proved in detail) become the centrepieces, both for theory and applications, and to provide concrete examples of stochastic differential equations used in finance. Finally, proofs of the existence, uniqueness and the Markov property of solutions of (general) stochastic equations complete the book.

Using careful exposition and detailed proofs, this book is a far more accessible introduction to Itô calculus than most texts. Students, practitioners and researchers will benefit from its rigorous, but unfussy, approach to technical issues. Solutions to the exercises are available online.

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Mastering Mathematical Finance

Mastering Mathematical Finance (MMF) is a series of short books that cover all core topics and the most common electives offered in Master's programmes in mathematical or quantitative finance. The books are closely coordinated and largely self-contained, and can be used efficiently in combination but also individually.

The MMF books start financially from scratch and mathematically assume only undergraduate calculus, linear algebra and elementary probability theory. The necessary mathematics is developed rigorously, with emphasis on a natural development of mathematical ideas and financial intuition, and the readers quickly see real-life financial applications, both for motivation and as the ultimate end for the theory. All books are written for both teaching and self-study, with worked examples, exercises and solutions.

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CAMBRIDGE UNIVERSITY PRESS

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Shaftesbury Road, Cambridge CB2 8EA, United Kingdom

One Liberty Plaza, 20th Floor, New York, NY 10006, USA

477 Williamstown Road, Port Melbourne, VIC 3207, Australia

314–321, 3rd Floor, Plot 3, Splendor Forum, Jasola District Centre, New Delhi – 110025, India

103 Penang Road, #05–06/07, Visioncrest Commercial, Singapore 238467

Cambridge University Press is part of Cambridge University Press & Assessment, a department of the University of Cambridge.

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Information on this title: www.cambridge.org/9781107002647

First published 2012

A catalogue record for this publication is available from the British Library

Library of Congress Cataloging-in-Publication data
Capinski, Marek, 1951–
Stochastic calculus for finance / Marek Capinski, Ekkehard Kopp, Janusz Traple.
p. cm. – (Mastering mathematical finance)
Includes bibliographical references and index.
ISBN 978-1-107-00264-7 (hardback : alk. paper) – ISBN 978-0-521-17573-9 (pbk. : alk. paper)
1. Finance – Mathematical models. 2. Stochastic processes.
3. Options (Finance) – Mathematical models.
I. Kopp, P. E., 1944– II. Traple, Janusz. III. Title.
HG106.C364 2012
332.01´51922 – dc23 2012024342

ISBN 978-1-107-00264-7 Hardback
ISBN 978-0-521-17573-9 Paperback

Additional resources for this publication at www.cambridge.org/9781107002647

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Preface

In this volume of the series 'Mastering Mathematical Finance' we develop the essential tools from stochastic calculus that will be needed in later volumes for the rigorous development of the Black–Scholes option pricing model and various of its extensions. Our motivation, and hence our choice of material, is again taken from the applications we have in mind: we develop only those parts of the theory that will be indispensable for the financial models discussed in this series. The Itô integral, with the Wiener process as its driving force, forms the heart of the text, with the Itô formula, developed in stages until we reach a sufficiently general setting, as the principal tool of our calculus.

The initial chapter sets the scene with an account of the basics of martingale theory in discrete time, and a brief introduction to Markov chains. The focus then shifts to continuous time, with a careful construction and development of the principal path, martingale and Markov properties of the Wiener process, followed by the construction of the Itô integral and discussion of its key properties. Itô processes are discussed next, and their quadratic variations are identified. Chapter 4 focuses on a complete proof of the Itô formula, which is often omitted in introductory texts, or presented as a by-product of more advanced treatments. The stringent boundedness assumptions required by an elementary treatment are removed by means of localisation, and the role of local martingales is emphasised. Applications of the Itô formula to the exponential martingale, the Feynman–Kac formula and integration by parts complete the chapter. The final chapter deals with existence and uniqueness of stochastic differential equations, motivated by the solution of the Black–Scholes equation and related examples.

The treatment throughout seeks to be thorough rather than comprehensive, and proofs are given in detail – sometimes deferred to the end of a chapter in order not to disrupt the flow of key ideas. The exercises form an integral part of the text; solutions and further exercises and solutions may be found at www.cambridge.org/9781107002647. Throughout, the reader is referred to the previous volumes in the series: to [DMFM] for initial motivation and to [PF] for basic results on measure and probability.

We wish to thank all who have read the drafts and provided us with feedback, especially Rostislav Polishchuk for very valuable comments.