Joseph Liouville
1809–1882
By permission of the
Académie des Sciences, Institut de France.
Dedicated to my granddaughter
ISABELLE SOFIE OLSEN
born January 11, 2009

and

to the memory of
JOSEPH LIOUVILLE
March 24, 1809–September 8, 1882
# Contents

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>Preface</td>
<td></td>
<td>xi</td>
</tr>
<tr>
<td>Notation</td>
<td></td>
<td>xiii</td>
</tr>
<tr>
<td>1</td>
<td>Joseph Liouville (1809–1882)</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Notes on Chapter 1</td>
<td>12</td>
</tr>
<tr>
<td>2</td>
<td>Liouville’s Ideas in Number Theory</td>
<td>13</td>
</tr>
<tr>
<td></td>
<td>Notes on Chapter 2</td>
<td>20</td>
</tr>
<tr>
<td>3</td>
<td>The Arithmetic Functions $\sigma_k(n)$, $\sigma_k^*(n)$, $d_{k,m}(n)$ and $F_k(n)$</td>
<td>24</td>
</tr>
<tr>
<td></td>
<td>Exercises 3</td>
<td>34</td>
</tr>
<tr>
<td></td>
<td>Notes on Chapter 3</td>
<td>41</td>
</tr>
<tr>
<td>4</td>
<td>The Equation $i^2 + jk = n$</td>
<td>43</td>
</tr>
<tr>
<td></td>
<td>Exercises 4</td>
<td>46</td>
</tr>
<tr>
<td></td>
<td>Notes on Chapter 4</td>
<td>47</td>
</tr>
<tr>
<td>5</td>
<td>An Identity of Liouville</td>
<td>48</td>
</tr>
<tr>
<td></td>
<td>Exercises 5</td>
<td>52</td>
</tr>
<tr>
<td></td>
<td>Notes on Chapter 5</td>
<td>53</td>
</tr>
<tr>
<td>6</td>
<td>A Recurrence Relation for $\sigma^*(n)$</td>
<td>54</td>
</tr>
<tr>
<td></td>
<td>Exercises 6</td>
<td>57</td>
</tr>
<tr>
<td></td>
<td>Notes on Chapter 6</td>
<td>58</td>
</tr>
<tr>
<td>7</td>
<td>The Girard-Fermat Theorem</td>
<td>59</td>
</tr>
<tr>
<td></td>
<td>Exercises 7</td>
<td>64</td>
</tr>
<tr>
<td></td>
<td>Notes on Chapter 7</td>
<td>65</td>
</tr>
</tbody>
</table>
Contents

8 A Second Identity of Liouville 67
Exercises 8 73
Notes on Chapter 8 76

9 Sums of Two, Four and Six Squares 77
Exercises 9 97
Notes on Chapter 9 98

10 A Third Identity of Liouville 100
Exercises 10 111
Notes on Chapter 10 114

11 Jacobi’s Four Squares Formula 116
Exercises 11 120
Notes on Chapter 11 121

12 Besge’s Formula 125
Exercises 12 133
Notes on Chapter 12 134

13 An Identity of Huard, Ou, Spearman and Williams 137
Exercises 13 155
Notes on Chapter 13 159

14 Four Elementary Arithmetic Formulae 163
Exercises 14 179
Notes on Chapter 14 182

15 Some Twisted Convolution Sums 184
Exercises 15 199
Notes on Chapter 15 201

16 Sums of Two, Four, Six and Eight Triangular Numbers 205
Exercises 16 220
Notes on Chapter 16 221

17 Sums of integers of the form $x^2 + xy + y^2$ 224
Exercises 17 232
Notes on Chapter 17 235

18 Representations by $x^2 + y^2 + z^2 + 2r^2$, $x^2 + y^2 + 2z^2 + 2r^2$
and $x^2 + 2y^2 + 2z^2 + 2r^2$ 239
Exercises 18 246
Notes on Chapter 18 249
Contents

19 Sums of Eight and Twelve Squares 251
   Exercises 19 258
   Notes on Chapter 19 259

20 Concluding Remarks 262

References 269
Index 283
In a series of eighteen papers published between the years 1858 and 1865 the French mathematician Joseph Liouville (1809–1882) introduced a powerful new method into elementary number theory. Liouville’s idea was to give a number of elementary (but not simple to prove) identities from which flowed many number-theoretic results by specializing the functions involved in the formulae.

Although Liouville’s ideas are now 150 years old, they still do not usually form part of a standard course in elementary number theory. Moreover there is no book in English devoted entirely to Liouville’s method, and, although some elementary number theory texts devote a chapter to Liouville’s ideas, most do not. In this book we hope to remedy this situation by providing a gentle introduction to Liouville’s method. We will not give a comprehensive treatment of all of Liouville’s identities but rather give a sufficient number of his identities in order to provide elementary arithmetic proofs of such number-theoretic results as the Girard-Fermat theorem, a recurrence relation for the sum of divisors function, Lagrange’s theorem, Legendre’s formula for the number of representations of a nonnegative integer as the sum of four triangular numbers, Jacobi’s formula for the number of representations of a positive integer as the sum of eight squares, and many others. We will also treat some of the more recent results that have been obtained using Liouville’s ideas.

Liouville’s method, although beautiful and arithmetic, is still an elementary one and as such has its limitations. As it is based on a number of identities, in order to obtain a particular number-theoretic result using it, the right identity has to be chosen as well as the right choice of the function occurring in it. And this is not always easy to do! Also, as with any elementary method, there are boundaries to what it can achieve. Indeed there are number-theoretic formulae which cannot be proved by Liouville’s method and other tools are required to prove them. However, on the other hand, although we do not know
Preface

the limitations of Liouville’s approach, there are still new number-theoretic
formulae waiting to be discovered and proved by Liouville’s method. Hopefully,
after reading this book, the reader will find some.

The prerequisites for this book include the basics of elementary number
theory such as divisibility, primes, the fundamental theorem of arithmetic,
quadratic reciprocity, the Legendre-Jacobi-Kronecker symbol, and a little about
the representation of integers by binary quadratic forms such as $x^2 + xy + y^2$,
$x^2 + y^2$ and $x^2 + 2y^2$. Hopefully in reading this book, the reader will enjoy
and appreciate the elegant arithmetic proofs that Liouville’s method enables us
to give. After reading this book the interested reader is encouraged to study the
theory of modular forms, where formulae similar to but deeper than the ones
given in this book can be found.

The author is grateful to his colleagues A. Alaca and S. Alaca for their
comments on the draft of this book, and to M. Huband for her help with
Chapter 1. The author is also grateful for the suggestions and corrections that he
received from B. C. Berndt of the University of Illinois. He also acknowledges
the kindness of Professor Berndt in allowing him to name this book in a
similar fashion to Berndt’s excellent book “Number Theory in the Spirit of
Ramanujan.” He also thanks his wife Carole for her help with the references
and index.

Kenneth S. Williams
Ottawa, Ontario, Canada
March 2010
Notation

\(\mathbb{N} = \) set of positive integers = \{1, 2, 3, \ldots\

\(\mathbb{N}_0 = \) set of nonnegative integers = \{0, 1, 2, 3, \ldots\

\(\mathbb{Z} = \) set of all integers = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\

\(\mathbb{Q} = \) set of all rational numbers

\(\mathbb{R} = \) set of all real numbers

\(\mathbb{C} = \) set of all complex numbers

\(\text{Re}(z) = \) real part of \(z \in \mathbb{C}\), that is \(\text{Re}(z) = x\), where \(z = x + iy, x, y \in \mathbb{R}\)

\(\text{Im}(z) = \) imaginary part of \(z \in \mathbb{C}\), that is \(\text{Im}(z) = y\), where \(z = x + iy, x, y \in \mathbb{R}\)

\(B_n = n\)-th Bernoulli number \((B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, \ldots)\)

\(d \mid n\) the integer \(d\) divides the integer \(n\)

\(d \nmid n\) the integer \(d\) does not divide the integer \(n\)

\(p^a \| n\) the prime \(p\) is such that \(p^a \mid n\) and \(p^{a+1} \nmid n\)

\(\gcd(m, n) = \) greatest common divisor of the integers \(m\) and \(n\) (not both zero), which we abbreviate to \((m, n)\) if space requires

\([x] = \) the greatest integer less than or equal to the real number \(x\)

\(\emptyset = \) the empty set

\(F_k(n) = \) \[\begin{cases} 1, & \text{if } k \mid n, \\ 0, & \text{if } k \nmid n. \end{cases}\)

\(G_2(\ell) = \) \[\begin{cases} 0, & \text{if } 2 \mid \ell, \\ 1, & \text{if } 2 \nmid \ell. \end{cases}\)

\(s(n) = \) \[\begin{cases} 1, & \text{if } n \text{ is a perfect square}, \\ 0, & \text{otherwise}. \end{cases}\)
\[ \sigma_k(n) = \begin{cases} \sum_{d \in \mathbb{N}} d^k, & \text{if } n \in \mathbb{N}, \\ 0, & \text{if } n \not\in \mathbb{N}. \end{cases} \]

\[ d(n) = \sigma_0(n) = \sum_{d \in \mathbb{N}} 1 = \text{number of positive divisors of } n \in \mathbb{N} \]

\[ \sigma(n) = \sigma_1(n) = \sum_{d \in \mathbb{N}} d = \text{sum of positive divisors of } n \in \mathbb{N} \]

\[ \sigma^*_k(n) = \sum_{d \in \mathbb{N}} d^k \]

\[ d^*(n) = \sigma^*_0(n) = \sum_{d \in \mathbb{N}} 1 \]

\[ \sigma^*(n) = \sigma^*_1(n) = \sum_{d \in \mathbb{N}} d \]

\[ d_{k,m}(n) = \sum_{d \in \mathbb{N}} 1 \]

\[ A(n) = \{(i, j, k) \in \mathbb{Z} \times \mathbb{N} \times \mathbb{N} | i^2 + jk = n, \ k \text{ odd}\} \]

\[ A_k(n) = \sum_{m \in \mathbb{N}} \sigma(m)\sigma(n - km) \]

\[ r_k(n) = \text{card}\{(x_1, \ldots, x_k) \in \mathbb{Z}^k | n = x_1^2 + \cdots + x_k^2\} \]

\[ p_k(n) = \text{card}\{(x_1, \ldots, x_k) \in \mathbb{Z}^k | n = x_1^2 + \cdots + x_k^2, \gcd(x_1, \ldots, x_k) = 1\} \]

\[ s_{2k}(n) = \text{card}\{(x_1, \ldots, x_{2k}) \in \mathbb{Z}^{2k} | n = x_1^2 + x_1x_2 + x_2^2 + \cdots + x_{2k-1}^2 + x_{2k-1}x_{2k} + x_{2k}^2\} \]

\[ t_k(n) = \text{card}\{(x_1, \ldots, x_k) \in \mathbb{N}_0^k | n = \frac{1}{2}x_1(x_1 + 1) + \cdots + \frac{1}{2}x_k(x_k + 1)\} \]

\[ r(n) = \text{card}\{(x, y) \in \mathbb{N}^2 | n = x^2 + y^2\} \]

\[ R_1(n) = \text{card}\{(x, y, z, t) \in \mathbb{N}^4 | n = x^2 + y^2 + z^2 + 2t^2\} \]

\[ R_2(n) = \text{card}\{(x, y, z, t) \in \mathbb{N}^4 | n = x^2 + y^2 + 2z^2 + 2t^2\} \]

\[ R_3(n) = \text{card}\{(x, y, z, t) \in \mathbb{N}^4 | n = x^2 + 2y^2 + 2z^2 + 2t^2\} \]
Notation

\[
\left( \frac{d}{n} \right) = \text{Legendre-Jacobi-Kronecker symbol, which is defined for } d \in \mathbb{Z} \text{ with } d \equiv 0, 1 \pmod{4} \text{ and } n \in \mathbb{N} (d \text{ is called the discriminant)}
\]

\[
\left( \frac{-3}{n} \right) = \begin{cases} 
+1, & \text{if } n \equiv 1 \pmod{3}, \\
-1, & \text{if } n \equiv 2 \pmod{3}, \\
0, & \text{if } n \equiv 0 \pmod{3}.
\end{cases}
\]

\[
\left( \frac{-4}{n} \right) = \begin{cases} 
+1, & \text{if } n \equiv 1 \pmod{4}, \\
-1, & \text{if } n \equiv 3 \pmod{4}, \\
0, & \text{if } n \equiv 0 \pmod{2}.
\end{cases}
\]

\[
\left( \frac{-7}{n} \right) = \begin{cases} 
+1, & \text{if } n \equiv 1, 2, 3 \pmod{7}, \\
-1, & \text{if } n \equiv 3, 5, 6 \pmod{7}, \\
0, & \text{if } n \equiv 0 \pmod{7}.
\end{cases}
\]

\[
\left( \frac{-8}{n} \right) = \begin{cases} 
+1, & \text{if } n \equiv 1 \text{ or } 3 \pmod{8}, \\
-1, & \text{if } n \equiv 5 \text{ or } 7 \pmod{8}, \\
0, & \text{if } n \equiv 0 \pmod{2}.
\end{cases}
\]

\[
\left( \frac{8}{n} \right) = \begin{cases} 
+1, & \text{if } n \equiv 1 \text{ or } -1 \pmod{8}, \\
-1, & \text{if } n \equiv 3 \text{ or } -3 \pmod{8}, \\
0, & \text{if } n \equiv 0 \pmod{2}.
\end{cases}
\]

\[
s(j, k) = \begin{cases} 
+1, & \text{if } j \equiv k \equiv 0 \pmod{2}, \\
-1, & \text{otherwise}.
\end{cases}
\]

\[
S_{e,f}(n) = \sum_{m=1}^{n-1} \sigma_e(m)\sigma_f(n-m)
\]

\[
T_{e,f,g}(n) = \sum_{m \in \mathbb{N}}^{m < n/g} \sigma_e(m)\sigma_f(n-gm)
\]

\[
W_{a,b}(n) := \sum_{m \in \mathbb{N}}^{m < n} \sigma(m)\sigma(n-m)
\]

\[
S(A, B, C, D, f; n) = \sum_{(a, b, x, y) \in \mathbb{N}^4}^{Cax + Dby = n} (f(Aa - Bb) - f(Aa + Bb))
\]
\[\mu(n) = \text{Möbius function} = \begin{cases} 1, & \text{if } n = 1, \\ (-1)^k, & \text{if } n = p_1 p_2 \ldots p_k, \text{ where } p_1, \ldots, p_k \text{ are distinct primes,} \\ 0, & \text{otherwise.} \end{cases}\]

\[\bar{\sigma}_s(n) := \sum_{d \in \mathbb{N}} (-1)^{d-1}d^s = \sigma_s(n) - 2^{s+1}\sigma_s(n/2)\]

\[\check{\sigma}_s(n) := \sum_{d \in \mathbb{N}} (-1)^{n/d-1}d^s = \sigma_s(n) - 2\sigma_s(n/2)\]

\[\bar{\sigma}(n) := \bar{\sigma}_1(n) = \sigma(n) - 4\sigma(n/2)\]

\[\check{\sigma}(n) := \check{\sigma}_1(n) = \sigma(n) - 2\sigma(n/2)\]

\[d_1(n) := \sum_{d \in \mathbb{N}} d = \sigma(n)\]

\[d_2(n) := \sum_{d \in \mathbb{N}} d = \sigma(n) - 2\sigma(n/2)\]

\[d_3(n) := \sum_{d \in \mathbb{N}} d = 2\sigma(n/2)\]

\[d_4(n) := \sum_{d \in \mathbb{N}} d = \sigma(n) - \sigma(n/2)\]

\[d_5(n) := \sum_{d \in \mathbb{N}} d = \sigma(n/2)\]

\[d_6(n) := \sum_{d \in \mathbb{N}} (-1)^{d-1}d = \sigma(n) - 4\sigma(n/2)\]

\[d_7(n) := \sum_{d \in \mathbb{N}} (-1)^{n/d-1}d = \sigma(n) - 2\sigma(n/2)\]

\[D(r, s; n) := \sum_{m=1}^{n-1} d_r(m)d_s(n - m)\]
\( \Delta := \text{set of triangular numbers} \)
\[ = \{0, 1, 3, 6, 10, 15, \ldots\} \]

\( R(n) := \text{card}\{(t_1, t_2, t_3, t_4) \in \Delta^4 \mid n = t_1 + t_2 + 2t_3 + 2t_4\} \)

\( \mathbb{H} := \{z \in \mathbb{C} \mid \text{Im}(z) > 0\} \)

\( SL_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}, \quad ad - bc = 1 \right\} \)

\( E_k(q) := 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n, \quad q = e^{2\pi iz}, \quad z \in \mathbb{H}, \quad k(\text{even}) \geq 2 \)

\( M_k(SL_2(\mathbb{Z})) := \text{space of modular forms of weight} \ k \ \text{for} \ SL_2(\mathbb{Z}) \)