CAMBRIDGE MONOGRAPHS ON
APPLIED AND COMPUTATIONAL
MATHEMATICS

Series Editors
M. ABLOWITZ, S. DAVIS, J. HINCH,
A. ISERLES, J. OCKENDON, P. OLVER

29 Partial Differential Equation
Methods for Image Inpainting
Cambridge Monographs on Applied and Computational Mathematics

The Cambridge Monographs on Applied and Computational Mathematics series reflects the crucial role of mathematical and computational techniques in contemporary science. The series publishes expositions on all aspects of applicable and numerical mathematics, with an emphasis on new developments in this fast-moving area of research.

State-of-the-art methods and algorithms as well as modern mathematical descriptions of physical and mechanical ideas are presented in a manner suited to graduate research students and professionals alike. Sound pedagogical presentation is a prerequisite. It is intended that books in the series will serve to inform a new generation of researchers.

A complete list of books in the series can be found at www.cambridge.org/mathematics.

Recent titles include the following:

12. Radial Basis Functions, Martin D. Buhmann
13. Iterative Krylov Methods for Large Linear Systems, Henk van der Vorst
14. Simulating Hamiltonian Dynamics, Benedict Leimkuhler & Sebastian Reich
16. Topology for Computing, Afra J. Zomorodian
17. Scattered Data Approximation, Holger Wendland
18. Modern Computer Arithmetic, Richard Brent & Paul Zimmermann
19. Matrix Preconditioning Techniques and Applications, Ke Chen
20. Greedy Approximation, Vladimir Temlyakov
22. The Mathematical Foundations of Mixing, Rob Sturman, Julio M. Ottino & Stephen Wiggins
23. Curve and Surface Reconstruction, Tamal K. Dey
24. Learning Theory, Felipe Cucker & Ding Xuan Zhou
25. Algebraic Geometry and Statistical Learning Theory, Sumio Watanabe
27. Difference Equations by Differential Equation Methods, Peter E. Hydon
28. Multiscale Methods for Fredholm Integral Equations, Zhongying Chen, Charles A. Micchelli & Yuesheng Xu
Partial Differential Equation Methods for Image Inpainting

CAROLA-BIBIANE SCHÖNLIEB

University of Cambridge

© in this web service Cambridge University Press
www.cambridge.org
To my love, Bertram
## Contents

<table>
<thead>
<tr>
<th>Preface</th>
<th>page ix</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>1 Introduction</strong></td>
<td></td>
</tr>
<tr>
<td>1.1 Digital Image Restoration in Modern Society</td>
<td>1</td>
</tr>
<tr>
<td>1.2 What is a Digital Image?</td>
<td>3</td>
</tr>
<tr>
<td>1.3 Image Inpainting</td>
<td>5</td>
</tr>
<tr>
<td><strong>2 Overview of Mathematical Inpainting Methods</strong></td>
<td></td>
</tr>
<tr>
<td>2.1 Variational and PDE Methods</td>
<td>10</td>
</tr>
<tr>
<td>2.2 Structure Versus Texture Inpainting</td>
<td>21</td>
</tr>
<tr>
<td>2.3 Inpainting of Colour Images</td>
<td>24</td>
</tr>
<tr>
<td>2.4 Video Inpainting</td>
<td>25</td>
</tr>
<tr>
<td><strong>3 The Principle of Good Continuation</strong></td>
<td></td>
</tr>
<tr>
<td>3.1 Gestalt Theory</td>
<td>27</td>
</tr>
<tr>
<td>3.2 Kanizsa’s Amodal Completion</td>
<td>29</td>
</tr>
<tr>
<td><strong>4 Second-Order Diffusion Equations for Inpainting</strong></td>
<td></td>
</tr>
<tr>
<td>4.1 An Axiomatic Approach to Image Inpainting</td>
<td>32</td>
</tr>
<tr>
<td>4.2 Harmonic Image Inpainting</td>
<td>41</td>
</tr>
<tr>
<td>4.3 Total Variation Inpainting</td>
<td>45</td>
</tr>
<tr>
<td>4.4 Absolutely Minimising Lipschitz Extensions</td>
<td>55</td>
</tr>
<tr>
<td>4.5 Further Reading and Some Extensions</td>
<td>59</td>
</tr>
<tr>
<td><strong>5 Higher-Order PDE Inpainting</strong></td>
<td></td>
</tr>
<tr>
<td>5.1 Second- Versus Higher-Order Approaches</td>
<td>63</td>
</tr>
<tr>
<td>5.2 Curvature-Based Inpainting</td>
<td>66</td>
</tr>
<tr>
<td>5.3 Cahn-Hilliard and TV-H$^{-1}$ Inpainting</td>
<td>85</td>
</tr>
<tr>
<td>5.4 Low Curvature Image Simplifiers</td>
<td>119</td>
</tr>
</tbody>
</table>
Contents

5.5 Second-Order Total Variation Inpainting 121
5.6 Further Reading and Some Extensions 133

6 Transport Inpainting 137
6.1 Inpainting by Transport Along Level Lines 137
6.2 Inpainting with Coherence Transport 144
6.3 GuideFill: Fast Artist-Guided Transport Inpainting 150

7 The Mumford-Shah Image Model for Inpainting 161
7.1 Inpainting with Mumford-Shah 161
7.2 Mumford-Shah-Euler Inpainting 170

8 Inpainting Mechanisms of Transport and Diffusion 174

9 Applications 180
9.1 Restoration of Medieval Frescoes 180
9.2 Road Reconstruction 189
9.3 Sinogram Inpainting for Limited Angle Tomography 191
9.4 Inpainting for 3D Conversion 204

Appendix A Exercises 211
Appendix B Mathematical Preliminaries 217
Appendix C MATLAB Implementation 229
Appendix D Image Credits 231

Glossaries 233
References 237
Index 253
Preface

The purpose of this book is to provide an introduction to the use of partial differential equations (PDEs) for digital image restoration. It is a way of sharing what I have learned while studying these methods for about ten years, defending a Ph.D. thesis on PDE inpainting, teaching courses in Göttingen and Cambridge on the topic and writing a couple of research papers on the use of PDEs in image inpainting along the way. Let me say what this book is and what it is not. It is:

1. An introduction to inpainting methods that use PDEs and local variational approaches to restore lost image contents;
2. An account from an enthusiast on some state-of-the-art inpainting methods from their abilities to their limitations and reference for informed researchers in the field of digital image processing; and
3. A work targeted at readers with basic knowledge in functional analysis, PDEs, measure theory and convex optimisation.

Therefore, I recommend this textbook only for students from the graduate level onwards.

This book is not:

1. A book that gives credit to the whole wealth of inpainting methods (in particular, this book will focus only on local inpainting methods and will scratch only non-local inpainting methods such as exemplar-based inpainting); or

One final comment before we go in medias res: the more I learn and understand about image inpainting, the more the complexity of the matter and
Preface

the variety of different problems, each requiring specialised methods, become apparent. Therefore, I feel a bit like Goethe’s Faust, who exclaims:

Habe nun, ach! Philosophie,
Juristerei und Medizin,
Und leider auch Theologie
Durchaus studiert, mit heißem Bemühn.
Da steh ich nun, ich armer Tor!
Und bin so klug als wie zuvor;
Heiße Magister, heiße Doktor gar
Und ziehe schon an die zehen Jahr
Herauf, herab und quer und krumm
Meine Schüler an der Nase herum –
Und sehe, daß wir nichts wissen können!
Das will mir schier das Herz verbrennen.

Let me finally mention some important people who have influenced me and this work. This book could not have been written without the support of many of my colleagues and friends.

First of all, I thank Bertram Düring, who accompanied and supported me throughout the work on this book. I also thank Peter Markowich – without him, I would never have started to write this book, nor finish it without his regular enquiries, ‘Und, wie gehts dem Buch?’ And I thank Arieh Iserles for inspiring me with his renowned book on the numerical solution of partial differential equations.

Many sections of this book are based on joint work with various co-authors: Wolfgang Baatz, Martin Benning, Andrea Bertozzi, Christoph Brune, Martin Burger, Luca Calatroni, Bertram Düring, Massimo Fornasier, Carsten Gottschlich, Lin He, Rob Hocking, Rien Lagerwerf, Andreas Langer, Jan Lellmann, Peter Markowich, Jean-Michel Morel, Stanley Osher, Kostas Papafitsoros and Bati Sengul. These people have made working on the topic of image inpainting interesting, exciting and fun.

I also owe many thanks to my image-analysis group for proofreading and invaluable comments on the presentation of this book. I especially thank Luca Calatroni, Rob Hocking, Jan Lellmann and Kostas Papafitsoros.