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978-1-009-19339-9 — The Calabi Problem for Fano Threefolds

C. Araujo, A.-M. Castravet, I. Cheltsov, K. Fujita, A.-S. Kaloghiros

J. Martinez-Garcia, C. Shramov, H. Süß, N. Viswanathan

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The Calabi Problem for Fano Threefolds

CAROLINA ARAUJO

Institute for Pure and Applied Mathematics (IMPA), Rio de Janeiro

ANA-MARIA CASTRAVET

University of Versailles

IVAN CHELTSOV

University of Edinburgh

KENTO FUJITA

Osaka University, Japan

ANNE-SOPHIE KALOGHIROS

Brunel University, London

JESUS MARTINEZ-GARCIA

University of Essex

CONSTANTIN SHRAMOV

Steklov Mathematical Institute, Moscow

HENDRIK SÜß

University of Jena, Germany

NIVEDITA VISWANATHAN

Loughborough University



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