Introduction

The Kähler–Einstein K-stability correspondence for Fano varieties is one of the most important contributions achieved in the 21st century [71, 212, 78, 82, 59, 214]. It links together complex algebraic geometry and analytic geometry:

a smooth Fano variety admits a Kähler–Einstein metric $\iff$ it is K-polystable.

However, the notion of K-stability is elusive and often difficult to check (see Chapter 1). On the other hand, for two-dimensional Fano varieties, Tian and Yau proved

Theorem ([215, 211]) *Let S be a smooth del Pezzo surface. Then S is K-polystable if and only if it is not a blow up of $\mathbb{P}^2$ in one or two points.*

Smooth Fano threefolds have been classified in [118, 119, 158, 159] into 105 families, which are labeled as №1.1, №1.2, №1.3, . . ., №9.1, №10.1 (see the Big Table in Chapter 6). Threefolds in each of these 105 deformation families can be parametrized by a non-empty irreducible rational variety [161, 163]. We pose the following problem.

Calabi Problem *Find all K-polystable smooth Fano threefolds in each family.*

This problem has already been solved for many families, and partial results are known in many cases [2, 3, 7, 14, 39, 46, 47, 55, 69, 79, 93, 101, 117, 146, 165, 199, 202, 212, 219, 227]. In particular, it has been proved in [93] that all smooth threefolds in the 26 families

№2.23, №2.28, №2.30, №2.31, №2.33, №2.35, №2.36, №3.14, №3.16, №3.18, №3.21, №3.22, №3.23, №3.24, №3.26, №3.28, №3.29, №3.30, №3.31, №4.5, №4.8, №4.9, №4.10, №4.11, №4.12, №5.2

are divisorially unstable (see Definition 1.20), so that none of them is K-polystable.

We show that all smooth Fano threefolds №2.26 are not K-polystable, and prove

Main Theorem *Let X be a general Fano threefold in the family №$\mathscr{N}$. Then*

$$X \text{ is K-polystable} \iff \mathscr{N} \neq 2.26 \text{ and } \mathscr{N} \notin \left\{\begin{array}{c} 2.23, 2.28, 2.30, 2.31, 2.33, \\ 2.35, 2.36, 3.14, 3.16, 3.18, \\ 3.21, 3.22, 3.23, 3.24, 3.26, \\ 3.28, 3.29, 3.30, 3.31, 4.5, \\ 4.8, 4.9, 4.10, 4.11, 4.12, 5.2 \end{array}\right\}.$$

Corollary *Let X be a general Fano threefold in the family №$\mathscr{N} \neq$ №2.26. Then*

$$X \text{ is K-polystable} \iff X \text{ is divisorially semistable} \iff X \text{ is K-semistable.}$$

Note that K-stability is an open property [170, 80, 19, 147]. Therefore, to prove that a general element of a given deformation family is K-polystable, it is enough to produce at least one K-stable (possibly singular) threefold in this family. However, this approach does not always work because many deformation families contain only Fano threefolds with infinite automorphism groups [45], so that none of these threefolds are K-stable, but some of them a priori could be K-polystable.

Before we finished the proof of the Main Theorem, its assertion had been already known for 65 deformation families (see Chapter 3 and Section 4.1 for more details). These families are

№1.1, №1.2, №1.3, №1.4, №1.5, №1.6, №1.7, №1.8, №1.10, №1.11,
№1.12, №1.13, №1.14, №1.15, №1.16, №1.17, №2.4, №2.23, №2.28,
№2.6, №2.29, №2.30, №2.31, №2.32, №2.33, №2.34, №2.35, №2.36,
№3.1, №3.11, №3.14,
№3.16, №3.18, №3.19, №3.20, №3.21, №3.22, №3.23, №3.24, №3.26,
№3.27, №3.28, №3.29, №3.30, №3.31, №4.4, №4.5, №4.7, №4.8, №4.9,
№4.10, №4.11, №4.12, №5.2, №5.3, №6.1, №7.1, №8.1, №9.1, №10.1.

For some families, we solved the Calabi Problem for all smooth threefolds in the family. For details, see the proof of the Main Theorem and check the Big Table in Chapter 6.

Example (see Section 4.7) Smooth Fano threefolds №2.24 are divisors in $\mathbb{P}^2 \times \mathbb{P}^2$ that have degree $(1, 2)$. For a suitable choice of coordinates $([x : y :$

$z], [u : v : w])$ on $\mathbb{P}^2 \times \mathbb{P}^2$, these smooth Fano threefolds can be described as follows.

(i) One parameter family that consists of threefolds given by

$$xu^2 + yv^2 + zw^2 + \mu\big(xvw + yuw + zuv\big) = 0, \qquad (\bigstar)$$

where $\mu \in \mathbb{C}$ such that $\mu^3 \neq -1$. All such threefolds are K-polystable.

(ii) One non-K-polystable threefold given by $(u^2+vw)x+(uw+v^2)y+w^2z = 0$,

(iii) One non-K-polystable threefold given by $(u^2 + vw)x + v^2y + w^2z = 0$.

If $\mu^3 = -1$ or $\mu = \infty$, then ($\bigstar$) defines a singular K-polystable Fano threefold.

Smooth Fano threefolds with infinite automorphism groups have been described in [45]. We completely solve the Calabi Problem for all of them. To be precise, we proved

Theorem *Let X be a smooth Fano threefold in the family №$\mathscr{N}$ such that $\mathrm{Aut}^0(X) \neq 1$. Then X is K-polystable if and only if either*

$$\mathscr{N} \in \begin{Bmatrix} 1.15, 1.16, 1.17, 2.20, 2.22, 2.27, 2.32, 2.34, 2.29, 3.5, 3.8, 3.9, 3.12, \\ 3.15, 3.17, 3.19, 3.20, 3.25, 3.27, 4.2, 4.3, 4.4, 4.6, 4.7, 4.13, 5.1, 5.3, \\ 6.1, 7.1, 8.1, 9.1, 10.1 \end{Bmatrix}$$

or one of the following cases hold:

- $\mathscr{N} = 1.10$ *and* $\mathrm{Aut}^0(X) \cong \mathrm{PGL}_2(\mathbb{C})$ *or* $\mathrm{Aut}^0(X) \cong \mathbb{G}_m$*;*
- $\mathscr{N} = 2.21$ *and* $\mathrm{Aut}^0(X) \cong \mathrm{PGL}_2(\mathbb{C})$ *or* $\mathrm{Aut}^0(X) \cong \mathbb{G}_m$*;*
- $\mathscr{N} = 2.24$ *and* $\mathrm{Aut}^0(X) \cong \mathbb{G}_m^2$*;*
- $\mathscr{N} = 3.10$ *and either* $\mathrm{Aut}^0(X) \cong \mathbb{G}_m^2$*, or* $\mathrm{Aut}^0(X) \cong \mathbb{G}_m$ *and X can be obtained by blowing up the smooth quadric threefold in $\mathbb{P}^4$ given by*

$$w^2 + xy + zt + a(xt + yz) = 0$$

along two conics that are given by $w^2 + zt = x = y = 0$ and $w^2 + xy = z = t = 0$, where $a \in \mathbb{C}$ is such that $a \notin \{0, \pm 1\}$, and x, y, z, t, w are coordinates on $\mathbb{P}^4$;
- $\mathscr{N} = 3.13$ *and* $\mathrm{Aut}^0(X) \cong \mathrm{PGL}_2(\mathbb{C})$ *or* $\mathrm{Aut}^0(X) \cong \mathbb{G}_m$*.*

At present, the Calabi Problem is not yet completely solved for the following 34 families:

№1.9, №1.10, №2.1, №2.2, №2.3, №2.4, №2.5, №2.6,
№2.7, №2.8, №2.9, №2.10, №2.11, №2.12, №2.13, №2.14,
№2.15, №2.16, №2.17, №2.18, №2.19, №2.20, №2.21, №2.22, №3.2,
№3.3, №3.4, №3.5, №3.6, №3.7, №3.8, №3.11, №3.12, №4.1.

For 27 of these families, we expect the following to be true:

Conjecture *All smooth Fano threefolds in the deformation families*

№1.9, №2.1, №2.2, №2.3, №2.4, №2.5, №2.6, №2.7, №2.8,
№2.9, №2.10, №2.11, №2.12, №2.13, №2.14, №2.15, №2.16, №2.17,
№2.18, №2.19, №3.2, №3.3, №3.4, №3.6, №3.7, №3.11, №4.1

are K-stable and, in particular, they are K-polystable.

The remaining seven families №1.10, №2.20, №2.21, №2.22, №3.5, №3.8, №3.12 contain non-K-polystable smooth Fano threefolds, but their general members are K-polystable. We present conjectural characterizations of their K-polystable members in Chapter 7.

Remark After the original version of this book appeared in June 2021 as Preprint 2021-31 in the preprint series of the Max Planck Institute for Mathematics, our Conjecture has been confirmed for the families №2.8, №3.3 and №4.1 in [16, 34, 145], and our conjectural characterizations of the K-polystable members of the families №2.22 and №3.12 have been proved in [38, 68]. Note that it follows from [38, 68] that every smooth Fano threefold in the deformation families №2.22 and №3.12 is K-semistable.

In Chapter 1, we present some K-stability results used in the proof of the Main Theorem. In Chapter 2, we prove the Tian–Yau theorem and find δ-invariants of del Pezzo surfaces. In Chapters 3, 4, and 5, we prove the Main Theorem. In Chapter 6, we present the Big Table that summarizes our results. In the Appendix, we present technical results used in the book.

Notations and conventions. Throughout this book, all varieties are assumed to be projective and defined over the field $\mathbb{C}$. For a variety X, we denote by $\overline{\mathrm{Eff}}(X)$, $\overline{\mathrm{NE}}(X)$ and $\mathrm{Nef}(X)$ the closure of the cone of effective divisors on X, the Mori cone of X, and the cone of nef divisors on X, respectively. For a subgroup $G \subset \mathrm{Aut}(X)$, we denote by $\mathrm{Cl}^G(X)$ and $\mathrm{Pic}^G(X)$ the subgroups in $\mathrm{Cl}(X)$ and $\mathrm{Pic}(X)$ consisting of Weil and Cartier divisors whose classes are G-invariant, respectively.

A subvariety $Y \subset X$ is said to be G-irreducible if Y is G-invariant and is not a union of two proper G-invariant subvarieties. We also denote by $\mathrm{Aut}(X,Y)$ the group consisting of automorphisms in $\mathrm{Aut}(X)$ that maps Y into itself.

We denote by $\mathbb{F}_n$ the Hirzebruch surface $\mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n))$. In particular, $\mathbb{F}_0 \cong \mathbb{P}^1 \times \mathbb{P}^1$, and the surface $\mathbb{F}_1$ is the blow up of $\mathbb{P}^2$ at a point.

For a divisor D on $\mathbb{P} = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_k}$, we say that D has degree $(a_1, a_2, \ldots, a_k)$ if

$$D \sim \sum_{i=1}^{k} \mathrm{pr}_i^* \big(\mathcal{O}_{\mathbb{P}^{n_i}}(a_i)\big),$$

where $\mathrm{pr}_i \colon \mathbb{P} \to \mathbb{P}^{n_i}$ is the projection to the ith factor. For a curve $C \subset \mathbb{P}$, we say that C has degree $(a_1, a_2, \ldots, a_k)$ if $\mathrm{pr}_i^*(\mathcal{O}_{\mathbb{P}^{n_i}}(1)) \cdot C = a_i$ for every $i \in \{1, \ldots, k\}$.

We denote by $\boldsymbol{\mu}_n$ the cyclic group of order n, we denote by D_{2n} the dihedral group of order $2n$, where $n \geqslant 2$ and $\mathrm{D}_4 = \boldsymbol{\mu}_2^2$. Similarly, we denote by $\mathfrak{S}_n$ and $\mathfrak{A}_n$ the symmetric group and its alternating subgroup, respectively. We denote by $\mathbb{G}_a$ the one-dimensional unipotent additive group, and we denote by $\mathbb{G}_m$ the one-dimensional algebraic torus.

We denote by $\mathbb{G}_m \rtimes \boldsymbol{\mu}_2$ the unique non-trivial semi-direct product of $\mathbb{G}_m$ and $\boldsymbol{\mu}_2$, we denote by $\mathbb{G}_m \rtimes \mathfrak{S}_3$ the unique non-trivial semi-direct product of $\mathbb{G}_m$ and $\mathfrak{S}_3$, and we denote by $\mathbb{G}_a \rtimes \mathbb{G}_m$ the semi-direct product such that $\mathbb{G}_m$ acts on $\mathbb{G}_a$ as $\mathbf{x} \mapsto t\mathbf{x}$.

For positive integers $n > k_1 > \cdots > k_r$, we denote by $\mathrm{PGL}_{n;k_1,\ldots,k_r}(\mathbb{C})$ the parabolic subgroup in $\mathrm{PGL}_n(\mathbb{C})$ that consists of images of matrices in $\mathrm{GL}_n(\mathbb{C})$ preserving a flag of subspaces of dimensions $k_1, \ldots, k_r$. For $n \geqslant 5$, we denote by $\mathrm{PSO}_{n;k}(\mathbb{C})$ the parabolic subgroup of $\mathrm{PSO}_n(\mathbb{C})$ preserving an isotropic linear subspace of dimension k. By $\mathrm{PGL}_{(2,2)}(\mathbb{C})$ we denote the image in $\mathrm{PGL}_4(\mathbb{C})$ of the group of block-diagonal matrices in $\mathrm{GL}_4(\mathbb{C})$ with two 2×2 blocks. This group acts on $\mathbb{P}^3$ preserving two skew lines. By $\mathrm{PGL}_{(2,2);1}(\mathbb{C})$ we denote the stabilizer in $\mathrm{PGL}_{(2,2)}(\mathbb{C})$ of a point on one of these lines.

Acknowledgments. We started this project in 2020 during a workshop on K-stability at the American Institute of Mathematics (San Jose, California, USA), which was organized by Mattias Jonsson and Chenyang Xu. We are very grateful to the Institute and the organizers for this workshop, which triggered our research.

Carolina Araujo was partially supported by CNPq and Faperj Research Fellowships. Ana-Maria Castravet was supported by the grant ANR-20-CE40-0023. Ivan Cheltsov was supported by EPSRC grant EP/V054597/1 and is also very grateful to the Max Planck Institute for Mathematics in Bonn for its hospitality. Kento Fujita has been partially supported by JSPS KAKENHI Grant Numbers 18K13388 and 22K03269. Anne-Sophie Kaloghiros was supported by an LMS Emmy Noether Fellowship and by EPSRC Grant EP/V056689/1. Jesus Martinez-Garcia was supported by EPSRC grant EP/V055399/1. Hendrik Süß was supported by EPSRC grants EP/V055445/1 and EP/V013270/1 and by the Carl Zeiss Foundation. Nivedita Viswanathan was partially supported by EPSRC Grant EP/V048619/1.

The authors would like to thank Hamid Abban (Ahmadinezhad), Harold Blum, Igor Dolgachev, Sir Simon Donaldson, Mattias Jonsson, Alexander Kuznetsov, Yuchen Liu, Yuji Odaka, Jihun Park, Andrea Petracci, Yuri Prokhorov, Sandro Verra, Chenyang Xu, Shing-Tung Yau and Ziquan Zhuang for many useful comments.

1
K-Stability

1.1 What is K-stability?

Let X be a Fano variety of dimension $n \geqslant 2$ that has Kawamata log terminal singularities. In most of the cases we consider, the variety X will be smooth. Set $L = -K_X$. A (normal) *test configuration* of the (polarized) pair $(X; L)$ consists of

- a normal variety $\mathcal{X}$ with a $\mathbb{G}_m$ action,
- a flat $\mathbb{G}_m$-equivariant morphism $p\colon \mathcal{X} \to \mathbb{P}^1$, where $\mathbb{G}_m$ acts naturally on $\mathbb{P}^1$ by
$$\big(t, [x : y]\big) \mapsto [tx : y],$$
- a $\mathbb{G}_m$-invariant p-ample $\mathbb{Q}$-line bundle $\mathcal{L} \to \mathcal{X}$ and a $\mathbb{G}_m$-equivariant isomorphism
$$\Big(\mathcal{X}\backslash p^{-1}(0), \mathcal{L}\big|_{\mathcal{X}\backslash p^{-1}(0)}\Big) \cong \Big(X \times \big(\mathbb{P}^1\backslash\{0\}\big), \mathrm{pr}_1^*(L)\Big),$$
where pr_1 is the projection to the first factor, and $0 = [0 : 1]$.

For such a test configuration, we let

$$\mathrm{DF}(\mathcal{X}; \mathcal{L}) = \frac{1}{L^n}\Big(\mathcal{L}^n \cdot K_{\mathcal{X}/\mathbb{P}^1} + \frac{n}{n+1}\mathcal{L}^{n+1}\Big). \tag{1.1}$$

This number is called *Donaldson–Futaki invariant* of the test configuration $(\mathcal{X}, \mathcal{L})$.

Remark 1.1 Quite often, we will omit $\mathcal{L}$ in $\mathrm{DF}(\mathcal{X}; \mathcal{L})$ and write it as $\mathrm{DF}(\mathcal{X})$.

Denote the central fiber $p^{-1}(0)$ by $\mathcal{X}_0$, and denote the fiber at infinity $p^{-1}(\infty)$ by $\mathcal{X}_\infty$, where $\infty = [1 : 0]$. The test configuration $(\mathcal{X}, \mathcal{L})$ is said to be

- *trivial* if there is a $\mathbb{G}_m$-equivariant isomorphism

$$\left(\mathcal{X}\backslash\mathcal{X}_\infty, \mathcal{L}\big|_{\mathcal{X}\backslash\mathcal{X}_\infty}\right) \cong \left(X \times \left(\mathbb{P}^1\backslash\infty\right), \mathrm{pr}_1^*(L)\right),$$

- *product-type* if we have an isomorphism $\mathcal{X}\backslash\mathcal{X}_\infty \cong X \times (\mathbb{P}^1\backslash\infty)$,
- *special* if the fiber $\mathcal{X}_0$ is irreducible, reduced, and $(\mathcal{X}, \mathcal{X}_0)$ has purely log terminal singularities, so that $\mathcal{X}_0$ is a Fano variety with Kawamata log terminal singularities.

Definition 1.2 The Fano variety X is said to be K-semistable if for every test configuration $(\mathcal{X}, \mathcal{L})$, $\mathrm{DF}(\mathcal{X}; \mathcal{L}) \geqslant 0$. Similarly, the Fano variety X is said to be K-stable if for every non-trivial test configuration $(\mathcal{X}, \mathcal{L})$, $\mathrm{DF}(\mathcal{X}; \mathcal{L}) > 0$. Finally, the Fano variety X is said to be K-polystable if it is K-semistable and

$$\mathrm{DF}(\mathcal{X}; \mathcal{L}) = 0 \iff (\mathcal{X}, \mathcal{L}) \text{ is of product type.}$$

Thus, we have the following implications:

$$X \text{ is K-stable} \Longrightarrow X \text{ is K-polystable} \Longrightarrow X \text{ is K-semistable.}$$

If X is not K-semistable, we say that X is K-unstable. Similarly, if X is K-semistable, but the Fano variety X is not K-polystable, we say that X is strictly K-semistable.

Theorem 1.3 ([6, 155]) *If X is K-polystable, then* $\mathrm{Aut}(X)$ *is reductive.*

Theorem 1.4 ([21, Corollary 1.3]) *If X is K-stable, then* $\mathrm{Aut}(X)$ *is finite.*

Corollary 1.5 *If* $\mathrm{Aut}(X)$ *is finite, then X is K-stable if and only if it is K-polystable.*

By the Chen–Donaldson–Sun theorem, the product of smooth K-polystable Fano varieties is K-polystable. This can be proved purely algebraically:

Theorem 1.6 ([225]) *Let V and Y be Fano varieties with Kawamata log terminal singularities. Then $V \times Y$ is K-semistable (resp. K-polystable, K-stable) if and only if V and Y are both K-semistable (resp. K-polystable, K-stable).*

Let G be a reductive subgroup in $\mathrm{Aut}(X)$. A given test configuration $(\mathcal{X}, \mathcal{L})$ is said to be G-equivariant if the product $G \times \mathbb{G}_m$ acts on $(\mathcal{X}, \mathcal{L})$ such that

- $\{1\} \times \mathbb{G}_m$ acting on $(\mathcal{X}, \mathcal{L})$ is the original $\mathbb{G}_m$-action,
- the $\mathbb{G}_m$-equivariant isomorphism

$$\left(\mathcal{X}\backslash p^{-1}(0), \mathcal{L}\big|_{\mathcal{X}\backslash p^{-1}(0)}\right) \cong \left(X \times (\mathbb{P}^1\backslash\{0\}), \mathrm{pr}_1^*(L)\right)$$

is $G \times \mathbb{G}_m$-equivariant.

Definition 1.7 The Fano variety X is said to be G-equivariantly K-polystable if for every G-equivariant test configuration $(\mathcal{X}, \mathcal{L})$, $\mathrm{DF}(\mathcal{X}; \mathcal{L}) \geqslant 0$, and $\mathrm{DF}(\mathcal{X}; \mathcal{L}) = 0$ if and only if $(\mathcal{X}, \mathcal{L})$ is of product type.

Remark 1.8 It has been proved in [142, 89] that it is enough to consider only special test configurations in Definitions 1.2 and 1.7.

If X is K-polystable, then X is G-equivariantly K-polystable. Surprisingly, we have

Theorem 1.9 ([67, 140, 148, 226]) *Suppose that X is G-equivariantly K-polystable. Then X is K-polystable.*

Remark 1.10 One can naturally define K-polystability for Fano varieties defined over an arbitrary field $\mathbb{F}$ of characteristic 0. By [226, Corollary 4.11], if X is defined over $\mathbb{F}$, and G is a reductive subgroup in $\mathrm{Aut}_{\mathbb{F}}(X)$, then

$$X \text{ is } G\text{-equivariantly K-polystable over } \mathbb{F} \iff X \text{ is K-polystable over } \overline{\mathbb{F}},$$

where $\overline{\mathbb{F}}$ is the algebraic closure of the field $\mathbb{F}$.

Let us conclude this section by briefly explaining how K-stability behaves in families.

Theorem 1.11 ([6, 19, 20, 80, 147, 141, 170, 218]) *Let $\eta\colon \mathcal{X} \to Z$ be a projective flat morphism such that $\mathcal{X}$ is $\mathbb{Q}$-Gorenstein, Z is normal, and all fibers of η are Fano varieties with at most Kawamata log terminal singularities. For every closed point $P \in Z$, let X_P be the fiber of the morphism η over P. Then the set*

$$\Big\{P \in Z \,\Big|\, X_P \text{ is K-stable}\Big\}$$

is a Zariski open subset of the variety Z. Similarly, the set

$$\Big\{P \in Z \,\Big|\, X_P \text{ is K-semistable}\Big\}$$

is a Zariski open subset of the variety Z. Furthermore, the set

$$\Big\{P \in Z \,\Big|\, X_P \text{ is K-polystable}\Big\}$$

is a constructible subset of the variety Z.

Thus, if X is a K-polystable smooth Fano threefold such that the group $\mathrm{Aut}(X)$ is finite, then X is K-stable by Corollary 1.5, so that general Fano threefolds in the deformation family of X are K-stable. We will use this observation often in the proof of the Main Theorem to prove that a general member of a given family is K-stable. Vice versa, to prove that a given Fano threefold is not K-polystable, we will use the following result (cf. [31, 170]).

Theorem 1.12 ([21, Theorem 1.1]) *Let $\eta\colon \mathcal{X} \to Z$ and $\eta'\colon \mathcal{X}' \to Z$ be projective surjective morphisms such that both $\mathcal{X}$ and $\mathcal{X}'$ are $\mathbb{Q}$-Gorenstein, Z is a smooth curve, and all fibers of η and η' are Fano varieties with at most Kawamata log terminal singularities. Let P be a point in Z, and let X_P and X'_P be the fibers of the morphisms η and η' over P, respectively. Suppose that there is an isomorphism $\mathcal{X} \setminus X_P \cong \mathcal{X}' \setminus X'_P$ that fits the following commutative diagram:*

$$\begin{tikzcd} \mathcal{X} \setminus X_P \arrow[r, "\cong"] \arrow[d, "\eta|_{\mathcal{X}\setminus X_P}"'] & \mathcal{X}' \setminus X'_P \arrow[d, "\eta'|_{\mathcal{X}'\setminus X'_P}"] \\ Z \setminus P \arrow[r, equal] & Z \setminus P \end{tikzcd}$$

If both X_P and X'_P are K-polystable, then they are isomorphic.

Together with Theorem 1.11, this result gives

Corollary 1.13 *Let $p\colon \mathcal{X} \to \mathbb{P}^1$ be a test configuration for the Fano variety X such that the fiber $p^{-1}(0)$ is a K-polystable Fano variety with at most Kawamata log terminal singularities that is not isomorphic to X. Then X is strictly K-semistable.*

In some cases, it is possible to prove that the general element of the deformation family of a K-polystable Fano threefold X is also K-polystable, even when X has infinite automorphism group. This is achieved by relating K-polystability and GIT polystability, an idea first investigated in [26, 206] in the analytic context. Suppose that X is a smooth K-polystable Fano variety of dimension n, and set $d = (-K_X)^n$. Let us briefly recall the setup of deformation theory; proofs and details can be found in [192, 153].

The infinitesimal deformation functor of the Fano variety X is denoted Def_X; recall that for an Artinian local $\mathbb{C}$-algebra A with residue field $\mathbb{C}$, $\mathrm{Def}_X(A)$ consists of isomorphism classes of commutative diagrams:

$$\begin{tikzcd} X \arrow[r, hook] \arrow[d] & \mathcal{X}_S \arrow[d] \\ \{0\}=\mathrm{Spec}(\mathbb{C}) \arrow[r, hook] & S=\mathrm{Spec}(A) \end{tikzcd}$$

An element $\{\mathcal{X}_S \to S\} \in \mathrm{Def}_X(A)$ is a deformation family of X over S. The tangent space of the deformation functor Def_X is $T^1_X = Ext^1(\Omega_X, \mathcal{O}_X)$ and $T^2_X = Ext^2(\Omega_X, \mathcal{O}_X)$ is an obstruction space for Def_X. As X is a smooth Fano, $T^1_X = H^1\big(X, \mathcal{T}_X\big)$ and $T^2_X = 0$ (deformations of X are unobstructed).