Mathematical Connections

A Companion for Teachers and Others
Mathematical Connections was developed at Education Development Center, Inc. (EDC) within the Center for Mathematics Education (CME) with partial support from the National Science Foundation.

This material is based on work supported by the National Science Foundation under Grant No. ESI-9617369. Any opinions, findings and conclusions or recommendations expressed in this material are those of the author(s) and do not necessarily reflect the views of the National Science Foundation.

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Education Development Center, Inc.
Newton, Massachusetts 02458

ISBN-13 978-0-88385-739-7 hardback
ISBN-10 0-88385-739-1 hardback

Library of Congress Catalog Control Number 200592349

Current Printing (last digit):
10 9 8 7 6 5 4 3 2 1
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A Companion for Teachers and Others

Al Cuoco

Center for Mathematics Education
Education Development Center, Inc.

Published and Distributed by
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Dedicated to the Memory of
Johnny Cuoco
1916–2003
Father, friend, and first baseman extraordinaire
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Introduction

This book grew out of our work at Education Development Center on a course for high school seniors as part of a curriculum called The CME Project [15]. That senior course provides students who have completed the equivalent of second-year high school algebra an exposure to some important ideas in classical and modern mathematics. More importantly, the course shows students some of the ways in which mathematicians work.

In developing The CME Project, my colleagues and I consulted regularly with high school teachers. Almost all of them told us that much of mathematics in the later years of the program was either new to them or buried deep in their undergraduate backgrounds. Together with these teachers, we came up with the idea for this book: a development of some mathematical topics and ideas, written specifically for practicing or prospective high school teachers. There are several things that this book is not:

- This is not a teaching guide for The CME Project. That curriculum already contains extensive solutions and teaching notes for the day-to-day teaching of the course.

- This is not a typical mathematics text. The focus on mathematical habits of mind leads to a different organization than the definition-theorem-proof-example style of writing. Instead, you’ll be led in on some of the ideas that lead up to the results, on different ways to think about topics, and on the connections among the topics. This is more about how mathematics is conceived than how it is presented. More about this approach in a minute.

- This is not a text for students. While many of the topics in the book might be appropriate for your classes, the translations are left to you.

- This is not a methods text. While the book is written with teachers in mind, the focus is on mathematics, not on pedagogy. If you are a teacher, you are the

Many curricula concentrate on important results in mathematics. The CME Project concentrates on important methods for obtaining those results.

About the sidenotes: Because this book emphasizes the thinking behind the results, the development is non-linear. Sidenotes give me a (less than perfect) way to evolve several ideas at once. You’ll develop your own style for dealing with them.

Mathematics has a tradition of presenting results in a logical and deductive style that shows how every fact can be deduced from previous ones. This is an extremely efficient, and often elegant, way of presenting one’s results. What’s missing from these presentations are the long periods of work that lead up to the final polished products. That’s what this book is about.
Introduction

expert on how to get ideas across to your students. You’ll find no discussions here about group work, alternative assessment, classroom uses of technology, or classroom dialogue. This book is for you as a mathematician.

What, then, is this book about?

It is about a closely knit collection of topics that are at the intersection of algebra, arithmetic, combinatorics, geometry, and calculus. It’s about some of the mathematics at the base of modern programs like The CME Project that mix discrete and continuous mathematics. It’s about some classical mathematics that has become more important (and tractable) because of advances in computational technology. And, most importantly, it is about some mathematical ways of thinking that I’ve found extremely useful in my high school teaching, both in my role as a mathematician and as a mentor for young people learning to do mathematics. I’ve picked out a suite of ideas that are a joy to my mind (and that I think you’ll enjoy), that will be useful in your teaching, that may be new to you, and that brings coherence to some of the topics you teach or will teach.

As a result, this is a rather personal book. You’re getting my choice of topics and a look at how a small circle of colleagues thinks about them. There are two implications:

- Other authors might have chosen other topics. I’ve become convinced, largely by working with other teachers over the past decade, of the value of depth (as opposed to breadth) in courses and books for teachers. Surveys have their place, but I wanted to write a book that takes a few simple themes—like fitting functions to data—and pushes them for all their worth.

- Other authors might have chosen other approaches. I make no apology for the fact that there’s a distinct bias towards algebra and algebraic thinking in most of the chapters. I’ve found it productive to focus on one thing at a time, and the particular approaches used in this book—things like abstracting regularity from repeated calculations—have been especially helpful in my own work and in my teaching.

The many contributors to the book have had high school mathematics teaching experience, and that, too, has guided the choice of topics and approach. For example, knowing about combinatorial proofs or difference calculus helps me design rich problem sets, field questions in class, connect ideas from one course to another, and know when a student’s idea is likely to grow into something that will be valuable for an entire class. But most mathematics courses in combinatorics, calculus, discrete mathematics, or number theory aren’t set up to develop applications to the teaching of high school mathematics. It’s not that they are bad courses—they have other purposes. As a result, most of what I know about these applications I’ve had to figure out for myself, learn from other teachers, or come upon by accident in books or articles. Many of my colleagues have had similar experiences and have tricks of the trade of their own. I’ve borrowed heavily from this collective wisdom. The book was designed for several settings:

- mathematics courses for prospective teachers,
- inservice courses for practicing teachers,
Introduction

- teacher study groups in a high school department,
- self study.

These purposes determined the book’s organization. Each chapter takes up one or two topics and develops them in depth. The heart of each chapter is the problem sets; as you probably tell your students, working the problems is where the fun is.

To facilitate self-study, the problem sets are rather orchestrated: they are meant to be done in sequence, and many of them refer to previous problems. In some cases, subsets of problems in a given set are chunked so that you know which problems “go together.” Many of the problems ask you to establish results that will be essential later in the book. If you skip any of these “essential” problems, you can go back to them when they are cited in later discussions. You’ll also find in the problem sets applications and extensions of the results in the chapter, previews of coming attractions, and connections to topics you teach.

At several points in the text, there are historical notes and references. A very good source for further historical readings is [3].

For too long, professional development for high school teachers has meant either taking yet another university mathematics course (with no connection to what we teach) or taking a workshop about teaching (with no mathematics in it at all). But the foundations of high school mathematics go deep enough to involve sophisticated thinking, hard problems, and subtle connections, all while staying connected to the kind of mathematics we talk about with our students every day. What I’ve attempted here is to outline one of many possible trips into these foundations of our subject. Enjoy.

Acknowledgements

The writing of this book was supported in part by NSF grant ESI 9617369. So many people contributed so many ideas that it’s impossible to list them all. But I have to mention a few:

- Michelle Manes, when she was my colleague at EDC, helped outline, critique, and edit earlier drafts of the book, and she did a significant part of the writing in Chapter 3. Her influence is everywhere.
- Ryota Matsuura, then a high school teacher in Brookline Massachusetts, worked through all the problems and wrote the hints and solutions that accompany each chapter. In addition, he found and helped fix mistakes, suggested improvements, and made everything much better.
- Wayne Harvey, director of our division at EDC, worked in detail through every problem in earlier drafts of Chapter 1, making specific suggestions for improving clarity and structure and pushing me to think about my goals for the book.

I’d also like to thank Steve Maurer, Peter Renz, and Barbara Hubbard for their very thoughtful reviews, Bowen Kerins and Ben Sinwell for trying some of the chapters in their course for teachers at the Park City Mathematics Institute, and
the thousands of high school students and teachers with whom I’ve been privileged to work over the past three decades.

Beverly Ruedi, Elaine Pedreira, and Don Albers from MAA were very generous with their vast expertise about publishing, design, and production. And my colleagues Helen Lebowitz and Nancy D’Amato at EDC pitched in at several important moments, creating graphics, transferring files, and staying in contact with MAA. All of these wonderful people make it all look easy.

Finally, none of this work (or anything else of value in my life) would have been possible without the love and support of my wife Micky.

— February 1, 2004
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Newton Massachusetts.
aicuoco@edc.org
Annotated Table of Contents

Chapter 1. Difference Tables and Polynomial Fits

Given a table like this:

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
</tr>
<tr>
<td>3</td>
<td>10</td>
</tr>
<tr>
<td>4</td>
<td>20</td>
</tr>
<tr>
<td>5</td>
<td>35</td>
</tr>
</tbody>
</table>

How do you find a simple (polynomial or exponential, say) function that agrees with the table? You may know that the old-fashioned theory of successive differences can be used:

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
<th>Δ</th>
<th>Δ²</th>
<th>Δ³</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>3</td>
<td>2</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>10</td>
<td>6</td>
<td>3</td>
<td>1</td>
</tr>
<tr>
<td>4</td>
<td>20</td>
<td>10</td>
<td>4</td>
<td>1</td>
</tr>
<tr>
<td>5</td>
<td>35</td>
<td>15</td>
<td>5</td>
<td>1</td>
</tr>
</tbody>
</table>

Here, Δ means difference. So, \(6 = 10 - 4, 5 = 15 - 10\), and so on. And Δ² means "second difference," the difference of the difference column.
And more generally, we'll see why a constant $r$th difference means an $r$th degree polynomial will work.

This is a standardized test maker's nightmare: there is a unique polynomial of minimal degree that agrees with a table, but there are infinitely many polynomials of higher degree that work, too. We'll see how to get them.

CAS is short for "Computer Algebra System." A CAS is a software environment that allows you to simplify, expand, and transform algebraic expressions. Examples are Mathematica and the CAS on the TI-89 calculator.

The multiplication rule: to multiply two complex numbers represented as vectors on the complex plane, multiply their lengths and add their arguments. If this sounds foreign to you, don't worry—we'll start from scratch in the chapter.

The third differences are constant, so there's a cubic polynomial that will agree with this data. This fact, without proof, is about as far as any current high school curriculum takes the subject.

In this chapter, we'll develop an extensive theory of finite differences:

- We'll look at why a constant third difference implies a cubic fit.
- We'll develop several different ways to find a polynomial that agrees with such a table. Some of these methods go back to Newton and Lagrange.
- We'll come up with a way to "fool" the table: classifying all polynomials that agree with a given set of points.

The methods we develop in chapter 1 will be essential throughout the book.

Chapter 2. Form and Function

This is a chapter about algebra. In precollege algebra, we're sort of cavalier about the nature of polynomials and algebraic expressions. When we work with graphs and equations—and when we use a graphing calculator—we're often thinking of polynomials as functions that describe variation or that produce outputs from inputs. When we go into "algebra mode" and discuss factoring, simplifying, and expanding—and when we use a CAS—we often treat polynomials as formal expressions, elements of an algebraic system that has an internal logic and arithmetic of its own. This chapter is about the connections between the two different ways to think about polynomials, and it's also about some classical algebra that's very useful in teaching but has been lost from most undergraduate courses. We'll see how some results of calculus, like the rule for taking the derivative of a polynomial, arise in a completely algebraic context and are forced on us by the rules of elementary arithmetic. In another direction, we'll generalize the "sum and product of the roots" topic from second-year algebra to more general relations between the roots and coefficients of a polynomial. These relations can be used to put the quadratic formula and the formula for a cubic equation into a more general setting.

Chapter 3. Complex Numbers, Complex Maps, and Trigonometry

We'll begin with an historical introduction to complex numbers, tracing their origin to the problem of finding roots of cubic equations. Then we'll look at the representation of complex numbers as points on the plane, and we'll give geometric interpretations for addition and multiplication. Our approach to the "multiplication rule" is quite simple. Most treatments use the addition formulas for sine and cosine. But in the summer of 2002, a group of high school teachers at PCMI (the Park City Mathematics Institute in Utah) discovered a proof of the rule that uses only elementary geometry. We'll look at their proof in this chapter.

The bonus of this approach is that one can turn the typical development on its head and use the multiplication rule to derive the addition formulas for sine and cosine. We'll do that, too.
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There’s a great debate these days about the role of trigonometric identities in high school mathematics. Are “identities” a topic that can safely be dropped from the curriculum? Is there any rhyme or reason to the pages of exercises that we (sometimes) ask students to do? Are there any really important identities for later work in science or mathematics? And how does technology impact what we can or should teach?

In this chapter, we take one path through this complex discussion. Two very important identities are the Pythagorean identity

$$\sin^2 x + \cos^2 x = 1$$

and the addition formulas

$$\sin(x + y) = \sin x \cos y + \cos x \sin y,$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y.$$  

The addition formulas lead to the double angle formulas:

$$\sin(2x) = 2\sin x \cos x,$$

$$\cos(2x) = 2\cos^2 x - 1.$$  

We’ll look at this small important collection and find similar formulas for \(\sin nx\) and \(\cos nx\) for any positive integer \(n\). This will involve creating a sequence of polynomials that shows up all over mathematics and science. This sequence brings coherence to trigonometry, and it connects trigonometry with arithmetic, algebra, and geometry.

We’ll next look at a key topic in many secondary programs: visualizing functions. You have several ways to visualize the function \(x \mapsto x^3\) when \(x\) runs over the real numbers. One of the most useful is a Cartesian graph of the function. But what if \(x\) is allowed to run over the complex numbers? How can you picture what’s going on? You can visualize the complex numbers as points on a plane. So, to get a “graph” of the function in the usual sense, you’d need four dimensions, and we don’t have that. But you can draw a picture of two complex planes (side by side, say), take a simple figure in one plane (maybe a circle) and draw its image (that is, the squares of all the complex numbers on your circle) in the other plane. By looking at the images of simple figures like lines and circles, you get a feeling for how the squaring map works on the complex numbers. This is just the beginning. By iterating certain functions, you get intricate and beautiful pictures. This area of iterated complex maps is at the base of many computer algorithms for generating life-like graphics.

Chapter 4. Combinations and Locks

There’s one aspect of combinatorics that has been extremely useful in my teaching: the notion of a combinatorial proof. Often, students notice a pattern in their work that can be expressed as an identity. For example, students often come up with the fact that the sum of the entries in any row of Pascal’s triangle is a power of 2.
Here, \( \binom{n}{k} \) means the entry in the \( k \)th slot of the \( n \)th row of Pascal’s triangle (where the numbering starts at 0).

If the details here are not perfectly clear to you, don’t worry. We’ll develop this example in detail later in the book.

There’s a legend involving the young Gauss that’s often attached to this formula. This means, for example, that

\[
1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n + 1)^2}{4}.
\]

Aren’t these beautiful formulas? Look at the way common factors show up. We’ll see in Chapter 5 how to look at the expressions in a way that makes them even more beautiful.

Chapter 5. Sums of Powers

You may remember that there’s a nice formula for the sum of the integers between 1 and \( n^2 \frac{n(n+1)}{2} \). You may also know that there are formulas for the sums of other powers. Here’s a list of the formulas for powers up to 8:

1. \( \frac{n(1 + n)}{2} \)
2. \( \frac{n(1 + n)(1 + 2n)}{6} \)
3. \( \frac{n^2(1 + n)^2}{4} \)
4. \( \frac{n(1 + n)(1 + 2n)(-1 + 3n + 3n^2)}{30} \)
5. \( \frac{n^2(1 + n)^2(-1 + 2n + 2n^2)}{12} \)
6. \( \frac{n(1 + n)(1 + 2n)(1 - 3n + 6n^3 + 3n^4)}{42} \)
7. \( \frac{n^2(1 + n)^2(2 - 4n - n^2 + 6n^3 + 3n^4)}{24} \)
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8. \[ \frac{n(n + 1)(1 + 2n)(-3 + 9n - n^2 - 15n^3 + 5n^4 + 15n^5 + 5n^6)}{90} \]

This chapter will investigate formulas like this—where they come from, how to generate them, and how to use them. For example, there’s a general recursive method for finding any one of these formulas. And there’s a method for generating them that comes right from the results of Chapter 1. In return, the formulas can be used to revisit the “finite differences” of Chapters 1 and 2, providing a particularly simple method for finding a polynomial that agrees with a table from its table of differences.

Prerequisites

I’d like to say that there are none, except for an interest in doing mathematics, and that’s almost true. But there are some things I’ll assume, at least for some of the chapters:

- You should have a pretty solid background in high school mathematics, including an acquaintance with Pascal’s triangle, the binomial theorem, proof by mathematical induction, and simple (arithmetic and geometric) series.
- In Chapter 2, it would be good if you know (or have once known) how to find the derivative of a polynomial and how to interpret the derivative as a formula for the slopes of tangents to its graph.
- In Chapter 3, you should know a little about complex numbers, the definitions of sine and cosine, and radian measure.
- In Chapter 4, we’ll occasionally need to multiply matrices, and, although we assume no specific prerequisites, some of the ideas will be connected to basic ideas in linear algebra.
- Chapter 5 requires the ideas from Chapters 1, 2, and 4, and little else.

The last three chapters are somewhat independent, and each depends on the first two.

Chapter 1 \(\rightarrow\) Chapter 2 \(\rightarrow\) Chapter 4

Chapter 3

Chapter 5

But please, don’t be scared off by all this. My advice is to dive into the chapters first and see if you can reconstruct the prerequisites when the need arises. If you need to consult a reference for some missing idea, you can find most of what you need in [15] and all of what you need in the additional references given at the end of each chapter.