Review of Calculus

An Outline of a Traditional Course in Calculus

Section I. Using the Limit Concept to Represent and Study the Major Concepts of Calculus for a Function of One Variable

1. How to Find Limits
   (a) Factor and cancel (when form is \( \frac{0}{0} \))
   (b) Use of a conjugate factor
   (c) Divide by highest power of \( x \) (when taking \( \lim_{x \to \infty} \) of a rational function)
   (d) Squeeze theorem
   (e) Know that \( \frac{0}{\pm} \) form means \( \pm \infty \)
   (f) L’Hospital’s rule (for 7 indeterminate forms)

2. How to Use Limits
   (a) To define continuity
      \[ \lim_{x \to c} f(x) = f(c) \]
   (b) To define the derivative
      \[ f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]
   (c) To determine types of discontinuities (removable, infinite, jump, oscillating)
   (d) To find asymptotes (vertical, horizontal, slant)
   (e) To evaluate an approximating sum for area of the form
      \[ \lim_{n \to \infty} \sum_{i=1}^{n} F(a + i \Delta x) \Delta x \]
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(f) To define the definite integral

\[ \int_a^b f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(c_i) \Delta x_i \]

(g) To evaluate improper integrals

\[ \int_a^\infty f(x) \, dx = \lim_{t \to \infty} \int_a^t f(x) \, dx \]

(h) To find the value for a convergent sequence

\[ \lim_{n \to \infty} a_n = A \]

3. How to Find Derivatives

(a) By the definition
(b) By standard formulas—power rule, product rule, quotient rule, chain rule
(c) Memorize formulas of standard functions and combine them with the above rules
(d) Method of implicit differentiation
(e) Method of logarithmic differentiation
(f) Formula for inverse functions

\[ \frac{d}{dx} f^{-1}(x) = \frac{1}{f'(y)} \]

(g) Formula for a function defined by an integral

\[ \frac{d}{dx} \int_a^x f(t) \, dt = f(x) \]

4. How to Use Derivatives

(a) To find tangent and normal lines to curves
(b) To find velocity and acceleration
(c) To find relative maximum and minimum points, inflection points, intervals of concavity
(d) To make an accurate graph for a function
(e) To solve max-min word problems
(f) To set up and solve related rate problems
(g) To solve differential equations (such as the one for exponential growth)
(h) To understand the algebraic and geometric meanings of the mean-value theorem
(i) To find approximations using differentials or Newton’s method
(j) To find limits of indeterminate forms using L’Hôpital’s rule
(k) To find polynomials that approximate a given curve
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5. How to Find Integrals
   (a) By the integral definition when the partition is into \( n \) equal parts
      \[
      \lim_{n \to \infty} \sum_{i=1}^{n} f(a + i \Delta x) \, \Delta x
      \]
   (b) By interpreting the integral as an area
   (c) By the fundamental theorem of calculus (using an antiderivative)
   (d) By a substitution which changes the integral to one we can integrate
   (e) By special methods of integration, such as integration by parts, partial fractions, and trigonometric substitutions
   (f) By use of limits to evaluate improper integrals
   (g) Find approximations using the trapezoid rule
   (h) Find approximations using Maclaurin series

6. How to Use Integrals
   (a) To find areas of regions by horizontal or vertical strips
   (b) To find volumes of solids of revolutions by discs, washers, or shells
   (c) To find the volume of a general solid by use of cross-sections
   (d) To find the length of a curve
   (e) To find the surface area of a solid of revolution
   (f) To find the center of mass of a region

7. How to Find and Use Values for Convergent Series
   (a) Finding patterns for sequences, and using the sequence of partial sums concept to define an infinite series
   (b) Methods to test series for convergence or divergence
      i. Comparison test
      ii. Integral test
      iii. Limit form of the comparison test
      iv. Ratio test
      v. Root test
      vi. Alternating series test
      vii. Absolute convergence test
   (c) Methods to find the value for a convergent series (exact or approximate)
      i. Geometric series
      ii. Telescoping principle
      iii. Use of an improper integral
      iv. Use of terms from the sequence of partial sums
   (d) The use of Maclaurin series of functions to find approximations for various expressions
Section II. Extending Calculus Results for a Function of One Variable to Functions of More than One Variable

8. Helpful Tools to Study Such Functions
   (a) Parametric representation of curves
   (b) Polar coordinates as an alternative to rectangular coordinates
   (c) Vectors in 2 and 3 dimensions, along with the dot product and cross product
   (d) Equations of curves and surfaces. The straight line is the simplest curve and the plane is the simplest of the quadric surfaces, which are the 3-dimensional generalization of conics from 2 dimensions.

9. Extensions of the Derivative
   (a) The partial derivative—the basic tool for calculations. It is used to express all the other concepts below.
   (b) Directional derivative (partial derivatives are a special case)
   (c) Chain rule and implicit differentiation
   (d) Gradient vector—a generalization of the normal line to a curve
   (e) Tangent plane—a generalization of the tangent line to a curve
   (f) Max-min values for surfaces

10. Extensions of the Integral
    (a) Definitions of a double and triple integral
    (b) Iterated integrals—the basic tool for evaluation of double and triple integrals
    (c) Use of polar coordinates—a generalization of the substitution method to evaluate a single integral
    (d) New ways to represent volumes of solids
    (e) New applications such as mass of a solid
    (f) Line integral concept, which generalizes the definite integral \( \int_a^b f(x) \, dx \)
    (g) Major integral theorems, such as Green’s theorem, Stokes’s theorem, and the divergence theorem, each of which can be considered a generalization of the fundamental theorem of calculus

11. How to Classify and Solve Differential Equations
    This topic comes after partial derivatives are introduced because the solution for an exact differential equation needs this concept. Otherwise, much of this topic could occur at the end of Section I.
    (a) Types of differential equations of the first order and first degree
        i. Separation of variables
        ii. Homogeneous
        iii. Integrating factors
        iv. Exact
        v. Linear
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(b) Higher-order linear homogeneous differential equations with constant coefficients

(c) Higher-order linear non-homogeneous differential equations with constant coefficients
   i. Method of undetermined coefficients
   ii. Method of variation of parameters

(d) Higher-order differential equations with non-constant coefficients
   i. Use a substitution to obtain a first-order, first-degree differential equation if possible
   ii. Find a series solution by solving for the coefficients of \( \sum_{n=1}^{\infty} a_n x^n \)

Section III. Precalculus Material that Is Needed for Calculus

12. Basic Algebraic Techniques
   (a) Factor polynomials and find zeros
   (b) Completing the square
   (c) Multiply by conjugate factor, i.e.,
      \[
      \frac{1}{\sqrt{a} - \sqrt{b}} = \frac{\sqrt{a} + \sqrt{b}}{a - b}
      \]
   (d) Use of the binomial expansion \((a + b)^n\)
   (e) A partial fraction decomposition, i.e.,
      \[
      \frac{2x + 1}{x(x - 1)^2} = \frac{A}{x} + \frac{B}{x - 1} + \frac{C}{(x - 1)^2}
      \]

13. Equations and Graphs of Familiar Curves
   (a) The straight line in two dimensions
   (b) The conics (parabola, circle, ellipse, and hyperbola)
   (c) Radical functions (i.e., \(\sqrt{x}, \sqrt[3]{x}\))
   (d) Rational functions, i.e.,
      \[
      \frac{2x + 1}{x - 2}
      \]
   (e) The six trigonometric functions
   (f) The inverse trigonometric functions, especially \(\arcsin x\) and \(\arctan x\)
   (g) The exponential and logarithmic functions \((y = b^x \text{ and } y = \log_b x)\)

14. Basic Trigonometric Identities
   (a) \(\sin^2 x + \cos^2 x = 1\)
   (b) \(\tan^2 x + 1 = \sec^2 x\)
   (c) \(\sin (x + y) = \sin x \cos y + \cos x \sin y\)
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\( \cos(x + y) = \cos x \cos y - \sin x \sin y \)

\( \sin 2x = 2 \sin x \cos x \)

\( \cos 2x = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1 = 1 - 2 \sin^2 x \)

\( \sin^2 x = \frac{1 - \cos 2x}{2}; \cos^2 x = \frac{1 + \cos 2x}{2} \)

\( \sin \frac{x}{2} = \sqrt{\frac{1 - \cos x}{2}}; \cos \frac{x}{2} = \sqrt{\frac{1 + \cos x}{2}} \)

15. Basic Logarithmic and Exponential Identities

(a) \( b^{x+y} = b^{x}b^{y} \)

(b) \( b^{x} = b^{x} \)

(c) \( b^{x-y} = \frac{b^{x}}{b^{y}} \)

(d) \( \log_{b}(xy) = \log_{b} x + \log_{b} y \)

(e) \( \log_{b} x^{r} = r \log_{b} x \)

(f) \( \log_{b} \left( \frac{x}{y} \right) = \log_{b} x - \log_{b} y \)

(g) \( \log_{b} x = \frac{\log_{a} x}{\log_{a} b} \)

(h) \( \log_{b} b^{x} = x \)

(i) \( b^{\log_{b} x} = x \)

16. Oft-used Formulas that are NOT True

(a) \( \sqrt{a} + \sqrt{b} = \sqrt{a + b} \)

(b) \( \frac{1}{a} + \frac{1}{b} = \frac{a + b}{ab} \)

(c) \( \sin 2x = 2 \sin x \)

(d) \( \log(x + y) = \log x + \log y \)

(e) \( \arcsin x = \frac{1}{\sin x} \)

Calculus Review Problems

Introduction

The best preparation for the study of calculus is a strong background in pre-calculus material, including such topics as algebraic techniques, trigonometric identities, Cartesian geometry, and properties of functions. This is because calculus adds new ideas—such as limits, derivatives, and integrals—to the material of pre-calculus mathematics, which allow us to obtain new insights. Solutions of most calculus problems contain a significant amount of pre-calculus material.

In a similar way, success in the study of real analysis depends strongly on a good understanding of calculus. Real analysis is deeply involved with theoretical aspects, yet it deals with the topics of limits, continuity, derivatives, integrals, and series that make up the subject matter of calculus. Most students begin their study of real analysis one year or more
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after completing their study of calculus and find their knowledge of and facility with calculus significantly diminished. For this reason, a review of calculus is excellent preparation for the real analysis course. The material below is intended to help you review the main topics from calculus that are needed for real analysis. Whereas they are divided into sections, some problems require results from more than one section. For example, the problems in Section 2 which use L’Hospital’s rule need derivative formulas from Section 3.

Most of the problems include comments to describe the importance of the problem or provide some hint for its solution. You may wish to attempt the problem first and look at the comments after your initial attempt.

Section 1. Pre-calculus Material

1. (a) Give a reason to explain why

\[ \sum_{i=1}^{n} i = \frac{n(n+1)}{2}. \]

(b) Prove the result in part (a) by the method of mathematical induction.

Comments: We use the summation formulas \( \sum_{i=1}^{n} i^k \) at the beginning of the chapter on integration in calculus to illustrate the integral definition for simple functions of the form \( f(x) = x^k \). In Example 5 of Problem 2 in Part II, the usual method for obtaining these formulas is described for the case when \( k = 2 \). It is a recursive approach, requiring knowledge of the formulas for \( \sum_{i=1}^{n} i^k \) for all natural numbers \( t < k \) in order to find the formula for \( \sum_{i=1}^{n} i^t \). It is clear that

\[ \sum_{i=1}^{n} i^0 = 1 + 1 + \cdots + 1 = n. \]

The formula

\[ \sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \]

is usually proved by writing

\[ S = 1 + 2 + \cdots + (n-1) + n \]

and also in the reverse order as

\[ S = n + (n-1) + \cdots + 2 + 1. \]

We obtain the desired formula by adding the two expressions together and simplifying.

2. Show that the slope of the line between the two points on the graph of \( f(x) = \sqrt{x} \) determined by \( x = a \) and \( x = b \) (where \( 0 < a < b \)) is equal to

\[ \frac{1}{\sqrt{a} + \sqrt{b}}. \]

This line is sometimes referred to as the secant line determined by \( x = a \) and \( x = b \).
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3. Nine familiar trigonometric identities are listed below. Assume that the first three are true, and use them to prove the next six.

\[ \sin^2 x + \cos^2 x = 1 \]  (1)
\[ \sin(x + y) = \sin x \cos y + \cos x \sin y \]  (2)
\[ \cos(x + y) = \cos x \cos y - \sin x \sin y \]  (3)
\[ \cos 2x = \cos^2 x - \sin^2 x \]  (4)
\[ \sin^2 x = \frac{1 - \cos 2x}{2} \]  (5)
\[ \cos^2 x = \frac{1 + \cos 2x}{2} \]  (6)
\[ \sin \frac{x}{2} = \sqrt{\frac{1 - \cos x}{2}} \]  (7)
\[ \tan^2 x + 1 = \sec^2 x \]  (8)
\[ \tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} \]  (9)

Comments: Trigonometric identities occur regularly in calculus. For example, identities (1) and (8) are used to simplify many calculations, identity (2) is used for the proof of the derivative formula for \( \sin x \), and identities (5) and (6) are used to evaluate integrals such as

\[ \int \sin^2 x \, dx \quad \text{and} \quad \int \cos^2 x \, dx. \]

4. Let \( t \) represent an angle using radian measure in a circle with radius 1 that is centered at the origin, as shown in Figure 1. It is clear from the graph that

Area of \( \triangle COA < \) Area of sector \( COB < \) Area of \( \triangle DOB. \)

![Figure 1](image-url)
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Substitute an expression for each of these three areas into this inequality and verify that it can be rewritten as

\[ \cos t < \frac{\sin t}{t} < \frac{1}{\cos t}. \]

Then explain how the squeeze theorem is used to prove that

\[ \lim_{t \to 0^+} \frac{\sin t}{t} = 1. \]

Comments: The result that

\[ \lim_{t \to 0} \frac{\sin t}{t} = 1 \]

is very important for the derivation of such calculus formulas as

\[ \frac{d}{dx} \sin x = \cos x. \]

All that is needed to prove

\[ \lim_{t \to 0} \frac{\sin t}{t} = 1 \]

are some simple trigonometric identities and the squeeze theorem.

Section 2. Limits and Continuity

5. (a) Verify that the function \( f(x) = x \sin(1/x) \) if \( x \neq 0 \) has a removable discontinuity at \( x = 0 \).

(b) Verify that the continuous function in part (a) (assuming that \( f(0) = 0 \)) is not differentiable at \( x = 0 \).

Comments: The result in part (a) can be verified by use of the squeeze theorem. The result in part (b) can be shown by use of the definition of \( f'(0) \).

6. Evaluate

\[ \lim_{h \to 0} \frac{\sqrt{2 + h} - \sqrt{2}}{h} \]

by the following three methods.

(a) Use a conjugate factor.

(b) Use L'Hopital's rule.

(c) Use the derivative of \( f(x) = \sqrt{2 + x} \), along with the definition of \( f'(0) \).

7. (a) Use L'Hopital's rule to evaluate

\[ \lim_{x \to \infty} (1 + \frac{a}{x})^{bx}. \]
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(b) How is the problem in part (a) related to the problem of finding the limit of the sequence \( \{x_n\} \) where

\[ x_n = \left(1 + \frac{a}{n}\right)^{bn}, \]

**Comments:** The limit in part (a) is indeterminate of the form 1\(^\infty\), so we first set

\[ y = \left(1 + \frac{a}{x}\right)^{bx} \]

and then rewrite this in the form

\[ \ln y = \frac{b \ln(1 + \frac{a}{x})}{x} \]

before applying L’Hospital’s rule. The function

\[ f(x) = \left(1 + \frac{a}{x}\right)^{bx} \]

in part (a) is sometimes referred to as the “related function” for the sequence in part (b), which has the same formula as the function in part (a), but is only defined for natural numbers \( n \). The same idea is used in the integral test for positive series. For example, to test the series

\[ \sum_{n=1}^{\infty} \frac{1}{n^2}, \]

consider the improper integral

\[ \int_1^{\infty} \frac{1}{x^2} \, dx. \]

In this case,

\[ f(x) = \frac{1}{x^2} \]

is the related function for the sequence \( \{a_n\} \) where

\[ a_n = \frac{1}{n^2}. \]

Section 3. Derivatives

8. Show how to use the formula

\[ \frac{d}{dx} (\cos x) = -\sin x \]

in order to obtain the usual derivative formulas for \( \sec x \) and \( \arccos x \).