MUSINGS OF THE MASTERS

An Anthology of Mathematical Reflections
To the memory of my parents, 
George and Nazeera, 
and my siblings 
Anne, Frank, and Lucien

I should like to express my deep gratitude and appreciation to Prof. Gerald Alexanderson. His outstanding scholarship, his patience, and his friendly persuasion have been vital in bringing this work to fruition.
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Introduction

This anthology is a collection of articles written by renowned mathematicians of the past century. The articles are on a variety of topics that, for want of a better name, shall be referred to as “humanistic.” An important criterion, thereby limiting the choice, is that the articles should be accessible to the general reader who need not have a technical knowledge of mathematics. These stipulations do not certify that the articles are easily understood! Nevertheless, it is hoped that the reader will find in them much of interest.

A motive in collecting these articles stems, in part, from a curiosity concerning the creative process in mathematics together with a curiosity concerning the essence of this remarkable subject. While some articles deal with this theme, it should quickly be said that these essays do not cast any decisive illumination on these matters!

Some of the essays have a common thread, while several others deal with a specific topic: Maak’s essay on Goethe and mathematics, and Severi’s article on Leonardo.

For the benefit of the non-specialist a few general remarks may be useful in placing the discipline of mathematics in a quasi-historical setting.

Mathematics has occupied a central position in the history of civilization. This is, in part, because of its antiquity, its universality, its utility, and its enigmatic dual nature. It spans divers cultures; on the one hand its dual nature allows it to function in a very wide spectrum of applications, and on the other hand in the “purest” of the arts and humanities. Over the years it has been revered by some and despised by others; it has been extolled by its advocates and condemned by its opponents. (St. Augustine, for example, was convinced that mathematicians were in league with the devil, but some scholars think that he conflated mathematics with astrology!) Such a discipline invites some comments.
Much has been written about the history of the subject and it is not necessary to review this in any detail. It is very likely that mathematics originated in the earliest civilizations as an empirical discipline designed to meet the practical needs of an organized society. It is equally likely that, given the inherent intellectual curiosity of some humans, it was not long thereafter that this innate curiosity stimulated a study of mathematics as a purely intellectual discipline. This natural inclination could account for the very remarkable reality that some mathematical facts were discovered, apparently independently from one another, in areas and cultures widely separated in space and in time. A notable example is the result known as the “Pythagorean Theorem” which was discovered independently not only by the ancient Greeks but by scholars on the Indian sub-continent as well as mathematicians in China.

Like other arts and sciences the progression of mathematics has not been continuous but has waxed and waned and the effect of cultural influences upon this variation has attracted the attention of some writers. There are periods of feverish creativity and fallow ones. As a side remark, it should be stressed that contemporary mathematics is experiencing a feverish, if not hectic, period of creativity.

Nor is it necessary to dwell on these matters, except to remind readers that about 600 B.C. in the hands of the Greeks, in what some have called the “golden age,” mathematics gradually became formalized. With occasional lapses, this formalization has continued to this day. As a result, mathematics has acquired a dual nature—that of a purely intellectual pursuit and that of a mechanism for coping with and modeling problems of the “real” world. Both of these facets have contributed to the intellectual history and the technological progress of mankind. For want of better descriptive terms, the theoretical pursuit will be referred to as the ‘sacred’ mode and the functional or practical side shall be called the ‘secular’ mode. Consumers of mathematics should not take offense at this choice of words. It is a fruitless pursuit to debate the relative merits of these modes since both have played, and continue to play, a significant role in societies—each has acted as a catalyst to the other.

In its sacred mode, the principal activities and principal objectives of mathematics can be described as the study of structures and the discernment of patterns. Although somewhat vague it would appear that this is as close as one can get to a focus or purpose of mathematical activity. In reality, however mathematics has apparently no well-defined objec-
tive. At any rate there has been no articulated objective upon which mathematicians have agreed. Different practitioners pursue their favorite structures with enthusiasm and sometimes recklessness! This is not to say that the subject lacks harmony and cohesion—it is a miraculous fact that in some mystical way, many of the various parts seem, eventually, to fit together into an organic whole. It should be said that the detection and discernment of this cohesion poses a constant challenge, especially to those who are fearful that the subject may disintegrate into an anarchy of mathematical feuds.

A mathematician, primarily in the sacred mode, is more or less free to define a set of axioms governing a structure and then draw inferences as to its properties. The mathematician is often free to search for a mathematical object with prescribed properties, however contrived these may seem to be. In reality there are subtle forces, not clearly understood, which prevent chaos from setting in.

Although there is a certain randomness in the choice of structure or property which is chosen for study, the history of mathematics shows that there is a ruthless selectivity that takes place, and in a manner that is far from being understood. The discipline casts off those parts that do not give it its proper nourishment. This selectivity is made, not by the practitioners but in a curious way, by the subject itself—the whole mathematical organism deprives the mathematician of his or her autonomous role, retaining only those parts that it deems essential. A hazardous activity then, is to try to predict the fate of a mathematical structure or object and one of the nightmares of a mathematician is to contemplate the future fate of his or her work! There is the story, possibly apocryphal, of Bertrand Russell’s dream in which he saw a phantom walking through the stacks of a library with a roaring fire below. The phantom examines a book taken from the shelf and either returns it to the shelf or casts it into the flames. Coming upon the *Principia*, a monumental work of Russell and Whitehead, the phantom pauses, hesitates for a moment—at which time Russell awoke from his dream!

In the secular mode, the mathematician or scientist when attempting to model a “real world” problem is constrained by the exigencies of the problem and cannot stray too far from the properties being modeled. To be sure, simplifying assumptions may, and often must, be made to render the problem accessible to whatever mathematical techniques are available. It has been repeatedly acknowledged that mathematics has been extraordinarily effective in modeling the real world. Structures or
mathematical constructs that have been studied as intellectual pursuits become, possibly after many decades, mental images of real world phenomena. The source and reason for this phenomenon is shrouded in mystery and has been the focus of much speculation but no satisfactory explanation has been forthcoming. Mathematics appears to permit a practitioner to conduct thought experiments, using different structures and different relations until some harmony with observation is achieved. It would appear as though in some unaccountable sense the mental constructs have mirrored the real world.

Although the distinction between secular and sacred is not sharply defined, it is in the sacred mode that mathematics most decisively exhibits its humanistic side. It is in this aspect of the subject that mathematicians, over the centuries, have stressed its aesthetic qualities that, on the face of it, appear paradoxical. For many it is an aesthetic experience perceived not through the senses, as one would normally expect for an aesthetic experience, but through the realm of thought. It is an art evoking powerful emotions but without the intervention of the senses.

Before continuing this train of thought, it is worthwhile pausing to make a distinction, again not too well defined. As in the arts, a distinction is made, however often blurred, between the popular and the serious. In the graphic arts for example, distinction is often made between so-called “commercial” and other forms of art and in the realm of poetry, for example, limericks are separated from serious poetry. To be sure some of these examples can become elevated to more significant levels. So in mathematics, there is a lighter side such as number puzzles and magic squares. Again some of these have found their way into the classics.

A few examples of mathematicians extolling the beauty of the subject can be noted. Writing to Sophie Germain, Gauss expressed himself as follows: “The enchanting charms of this sublime subject do not reveal themselves except to those who have the courage to plumb its depths.” Poincaré wrote: “We are astonished to find the senses involved in mathematical proofs which should appeal only to the intellect. There is...a sense...of beauty of the harmony of number and form, of geometric elegance. It is a genuine aesthetic sense which all mathematicians know.” From the pen of Russell we read “mathematics rightly understood, possesses not only truth but supreme beauty.” These examples could be multiplied tenfold but we end with an anonymous quote: “When we read the memoirs of Gauss ... does not the overall structure recall one of the marvelous temples ... raised to the Hellenic divinities.”
In these and other cases, the beauty and harmony are palpable but not through the ordinary senses.

Another question that arises is that of the reality of mathematics—where does it reside? The issue has provoked much discussion and debate, often pitting one side against another; its ultimate resolution is far from certain. Indeed, the point of view one takes is as much a matter of quasi-theological belief as it is a matter of intellectual certitude.

Some hold that its reality resides entirely internally, that is, that mathematics is created as a musician might create a quartet or an artist paint a Madonna. Others say that it is entirely external—that mathematics is a process of discovery. Not surprisingly others take a middle of the road view, that it is a combination of the two. A well-known proponent of this last view is Kronecker who in his famous dictum wrote: “God made the integers—all else is man made.” G.H. Hardy is an advocate of the second view: “We believe,” he wrote, “that mathematical reality lies outside of us, that our function is to discover or observe it and the theorems which we grandiloquently describe as our own creations, are simply our notes of our observations.” Hardy’s view is supported by Hermite who wrote: “There exists, if I am not mistaken, a whole world which is the set of all mathematical truths to which we have access by intelligence, as there exists a world of reality—both independent of us and both divine creations.” Hilbert represents the other extreme writing “mathematics is a meaningless formal game.” And the debate continues.

To be sure the discipline “philosophy of mathematics,” a time-honored pursuit, has flourished in recent years. It is difficult, however, for one uninitiated, to find enlightenment here. The reason is that much of the discourse is phrased in technical philosophical vocabulary—terms which are remote from the layperson’s experience. Such terms as “Platonism,” “Cartesian duality,” “epistemology,” Hegelianism, post modernism, existentialism, and so on, are alien to most laypersons’ experiences.

In view of its elusive and recondite nature, one might expect that mathematicians would rise to the challenge and write more extensively about their subject. This is, surprisingly, not the case. With some notable historical exceptions, such as Descartes, Leibniz, and in more recent decades Brouwer, Poincaré, Hadamard, Hardy, and Weyl, mathematicians have been curiously reluctant to write on matters that lie outside the strict confines of mathematics proper and mathematical
research. It is not easy to account for this reluctance but some contribut-
ing factors suggest themselves.

First, writing about mathematics and areas contiguous to it, is a
somewhat daunting prospect, for it implies that we know what we are
talking about. As noted above the nature of the subject is elusive, and
this elusiveness has been widely recognized by mathematicians. There
have been many attempts to define mathematics and these are so unsat-
sifying that this writer’s favorite is that of Bacon who wrote that “math-
ematics is the study that makes men subtile (sic)”! This is amusing but
hardly enlightening.

Second, mathematicians by their training and practice have become
highly immersed in habits of precise thought and reasoning in their dis-
cipline. To write about other subjects—ethics, religion, politics—is a
dismaying prospect, for these subjects are by their nature imprecise and
a discourse requires a transition into different modes of thought, a tran-
sition that is not easy to make. Many mathematicians have, nevertheless,
vented very successfully into areas such as philosophy, and fiction.

There is another factor that influences this inhibition and reticence.
This has to do with the culture of the mathematical community. The tra-
ditions of the past were somewhat rigid and these same traditions
viewed the writing of textbooks or popular works on mathematics with
a slight degree of scorn. Such writing was well received after the writer
had established a reputation as a research scholar. Many examples may
be cited.

As a result of these and other induced inhibitions, society has been
largely deprived of the musings of first-rate mathematical minds—a
depprivation that is unfortunate. Some few courageous souls however,
possibly coming initially from different traditions, have set pen to paper
and have left a worthwhile legacy. To be sure a gift in one area of the
intellect does not necessarily carry over to other areas—how else can
we account for the outrageous excesses of which some mathematicians
have been guilty. Fortunately however, there has been a moderate
amount of transfer and the fruits of this transfer are articles by promi-
nent mathematicians writing on topics sometimes contiguous to, and
sometimes remote from, their principal occupation—that of mathemati-
cal research.

These articles have been assembled and are offered in a single place.
They span roughly a century in time and a wide range in subject. They
are by mathematicians acknowledged by their peers as outstanding cre-
Introduction

Ators whose work has added richly to the discipline. Each article is preceded by a brief biographical sketch of the author and a short indication of the content. Many of them elaborate on some of the issues that have been raised above. To be sure in many articles there is a common theme—the attempt to define the role of intuition in mathematical activities—intuition as contrasted with pure formalism, a contrast that can be briefly, if too simply, described by asking whether mathematics is invented or discovered.

We hope that the appearance of these articles in a single place will encourage mathematicians to share with others their perceptions on this beautiful subject, or on other matters dealing more directly with the humanistic side of our nature.
Contents

Introduction ........................................ vii

Part I: Mathematics and the Intellect
Mathematics and Thinking Mathematically, Mary Cartwright . . . 3
Mathematical Invention, Henri Poincaré ............................ 17
Thoughts on the Heuristic Method, Jacques Hadamard .......... 31
Mathematical Proof, G. H. Hardy ............................... 45
The Unity of Knowledge, Hermann Weyl .......................... 65

Part II: Mathematics and Human Understanding
Mathematics and the Arts, Marston Morse .......................... 81
Intuition, Reason and Faith in Science, George David Birkhoff . . 95
Logic and the Understanding of Nature, David Hilbert ............ 115
The Cultural Basis of Mathematics, Raymond Wilder ............ 129

Part III: Mathematics and Society
Presidential Address to the British Association, J.J. Sylvester . . 151
The Mathematician, John von Neumann ................................ 169
The Community of Scholars, André Lichnerowicz ................. 185
History of Mathematics: Why and How, André Weil .............. 199

Part IV: Miscellaneous
Does God Exist?, Paul Lévy ....................................... 217
Goethe and Mathematics, Wilhelm Maak .......................... 231
Leonardo and Mathematics, Francesco Severi .................... 247
The Highest Good, Norbert Wiener .............................. 257

xv
Sources of the Articles ........................................ 271
Index .......................................................... 273