PROBLEMS

THE TWENTY-SIXTH WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

November 20, 1965

A-1. Let ABC be a triangle with angle $A <$ angle $C < 90° <$ angle B. Consider the bisectors of the external angles at A and B, each measured from the vertex to the opposite side (extended). Suppose both of these line-segments are equal to AB. Compute the angle A.

A-2. Show that, for any positive integer n,

$$\sum_{r=0}^{[(n-1)/2]} \left\{\frac{n-2r}{n}\binom{n}{r}\right\}^2 = \frac{1}{n}\binom{2n-2}{n-1},$$

where $[x]$ means the greatest integer not exceeding x, and $\binom{n}{r}$ is the binomial coefficient "n choose r," with the convention $\binom{n}{0}=1$.

A-3. Show that, for any sequence $a_1, a_2, \cdots$ of real numbers, the two conditions

(A)
$$\lim_{n\to\infty} \frac{e^{(ia_1)} + e^{(ia_2)} + \cdots + e^{(ia_n)}}{n} = \alpha$$

and

(B)
$$\lim_{n\to\infty} \frac{e^{(ia_1)} + e^{(ia_4)} + \cdots + e^{(ia_{n^2})}}{n^2} = \alpha$$

are equivalent.

A-4. At a party, assume that no boy dances with every girl but each girl dances with at least one boy. Prove that there are two couples gb and $g'b'$ which dance whereas b does not dance with g' nor does g dance with b'.

A-5. In how many ways can the integers from 1 to n be ordered subject to the condition that, except for the first integer on the left, every integer differs by 1 from some integer to the left of it?

A-6. In the plane with orthogonal Cartesian coordinates x and y, prove that the line whose equation is $ux+vy=1$ will be tangent to the curve $x^m+y^m=1$ (where $m>1$) if and only if $u^n+v^n=1$ and $m^{-1}+n^{-1}=1$.

B-1. Evaluate

$$\lim_{n\to\infty} \int_0^1 \int_0^1 \cdots \int_0^1 \cos^2\left\{\frac{\pi}{2n}(x_1+x_2+\cdots x_n)\right\} dx_1 dx_2 \cdots dx_n.$$

B-2. In a round-robin tournament with n players $P_1, P_2, \cdots, P_n$ (where $n>1$), each player plays one game with each of the other players and the rules are such that no ties can occur. Let ω_r and l_r be the number of games won and lost, respectively, by P_r. Show that

$$\sum_{r=1}^{n} w_r^2 = \sum_{r=1}^{n} l_r^2.$$

B-3. Prove that there are exactly three right-angled triangles whose sides are integers while the area is numerically equal to twice the perimeter.

B-4. Consider the function

$$f(x, n) = \frac{\binom{n}{0} + \binom{n}{2}x + \binom{n}{4}x^2 + \cdots}{\binom{n}{1} + \binom{n}{3}x + \binom{n}{5}x^2 + \cdots},$$

where n is a positive integer. Express $f(x, n+1)$ rationally in terms of $f(x, n)$ and x. Hence, or otherwise, evaluate $\lim_{n\to\infty} f(x, n)$ for suitable fixed values of x. (The symbols $\binom{n}{r}$ represent the binomial coefficients.)

B-5. Consider collections of unordered pairs of V different objects $a, b, c, \cdots, k$. Three pairs such as bc, ca, ab are said to form a triangle. Prove that, if $4E \leqq V^2$, it is possible to choose E pairs so that no triangle is formed.

B-6. If A, B, C, D are four distinct points such that every circle through A and B intersects (or coincides with) every circle through C and D, prove that the four points are either collinear (all of one line) or concyclic (all on one circle).

THE TWENTY-SEVENTH WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

November 19, 1966

A-1. Let $f(n)$ be the sum of the first n terms of the sequence $0, 1, 1, 2, 2, 3, 3, 4, \cdots$, where the nth term is given by

$$a_n = \begin{cases} n/2 & \text{if } n \text{ is even,} \\ (n-1)/2 & \text{if } n \text{ is odd.} \end{cases}$$

Show that if x and y are positive integers and $x>y$ then $xy=f(x+y)-f(x-y)$.

A-2. Let a, b, c be the lengths of the sides of a triangle, let $p=(a+b+c)/2$, and r be the radius of the inscribed cricle. Show that

$$\frac{1}{(p-a)^2}+\frac{1}{(p-b)^2}+\frac{1}{(p-c)^2} \geqq \frac{1}{r^2}.$$

A-3. Let $0<x_1<1$ and $x_{n+1}=x_n(1-x_n)$, $n=1, 2, 3, \cdots$. Show that

$$\lim_{n\to\infty} nx_n = 1.$$

A-4. Prove that after deleting the perfect squares from the list of positive integers the number we find in the nth position is equal to $n+\{\sqrt{n}\}$, where $\{\sqrt{n}\}$ denotes the integer closest to $\sqrt{n}$.

A-5. Let C denote the family of continuous functions on the real axis. Let T be a mapping of C into C which has the following properties:

1. T *is linear*, i.e. $T(c_1\psi_1+c_2\psi_2)=c_1T\psi_1+c_2T\psi_2$, for c_1 and c_2 real and ψ_1 and ψ_2 in C.
2. T *is local*, i.e. if $\psi_1\equiv\psi_2$ in some interval I then also $T\psi_1\equiv T\psi_2$ holds in I.

Show that T must necessarily be of the form $T\psi(x)=f(x)\psi(x)$ where $f(x)$ is a suitable continuous function.

A-6. Justify the statement that

$$3=\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+5\sqrt{1+\cdots}}}}}.$$

B-1. Let a convex polygon P be contained in a square of side one. Show that the sum of the squares of the sides of P is less than or equal to 4.

B-2. Prove that among any ten consecutive integers at least one is relatively prime to each of the others.

B-3. Show that if the series

$$\sum_{n=1}^{\infty}\frac{1}{p_n}$$

is convergent, where $p_1, p_2, p_3, \cdots, p_n, \cdots$ are positive real numbers, then the series

$$\sum_{n=1}^{\infty} \frac{n^2}{(p_1 + p_2 + \cdots + p_n)^2} p_n$$

is also convergent.

B-4. Let $0 < a_1 < a_2 < \cdots < a_{mn+1}$ be $mn+1$ integers. Prove that you can select either $m+1$ of them no one of which divides any other, or $n+1$ of them each dividing the following one.

B-5. Given $n(\geqq 3)$ distinct points in the plane, no three of which are on the same straight line, prove that there exists a simple closed polygon with these points as vertices.

B-6. Show that all solutions of the differential equation $y'' + e^x y = 0$ remain bounded as $x \to \infty$.

THE TWENTY-EIGHTH WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

December 2, 1967

A-1. Let $f(x)=a_1\sin x+a_2\sin 2x+\cdots+a_n\sin nx$, where $a_1, a_2, \cdots, a_n$ are real numbers and where n is a positive integer. Given that $|f(x)|\leqq|\sin x|$ for all real x, prove that

$$|a_1+2a_2+\cdots+na_n|\leqq 1.$$

A-2. Define S_0 to be 1. For $n\geqq 1$, let S_n be the number of $n\times n$ matrices whose elements are nonnegative integers with the property that $a_{ij}=a_{ji}$, $(i,j=1, 2, \cdots, n)$ and where $\sum_{i=1}^{n} a_{ij}=1$, $(j=1, 2, \cdots, n)$. Prove

(a) $S_{n+1}=S_n+nS_{n-1}$,

(b) $\displaystyle\sum_{n=0}^{\infty} S_n\frac{x^n}{n!}=\exp(x+x^2/2)$, where $\exp(x)=e^x$.

A-3. Consider polynomial forms ax^2-bx+c with integer coefficients which have two distinct zeros in the open interval $0<x<1$. Exhibit with a proof the least positive integer value of a for which such a polynomial exists.

A-4. Show that if $\lambda>\frac{1}{2}$ there does not exist a real-valued function u such that for all x in the closed interval $0\leqq x\leqq 1$, $u(x)=1+\lambda\int_x^1 u(y)\,u(y-x)\,dy$.

A-5. Show that in a convex region in the plane whose boundary contains at most a finite number of straight line segments and whose area is greater than $\pi/4$ there is at least one pair of points a unit distance apart.

A-6. Given real numbers $\{a_i\}$ and $\{b_i\}$, $(i=1, 2, 3, 4)$, such that $a_1b_2-a_2b_1\neq 0$. Consider the set of all solutions (x_1, x_2, x_3, x_4) of the simultaneous equations

$$a_1x_1+a_2x_2+a_3x_3+a_4x_4=0 \text{ and } b_1x_1+b_2x_2+b_3x_3+b_4x_4=0,$$

for which no x_i $(i=1, 2, 3, 4)$ is zero. Each such solution generates a 4-tuple of plus and minus signs (signum x_1, signum x_2, signum x_3, signum x_4).

(a) Determine, with a proof, the maximum number of distinct 4-tuples possible.

(b) Investigate necessary and sufficient conditions on the real numbers $\{a_i\}$ and $\{b_i\}$ such that the above maximum number of 4-tuples is attained.

B-1. Let $(ABCDEF)$ be a hexagon inscribed in a circle of radius r. Show that if $\overline{AB}=\overline{CD}=\overline{EF}=r$, then the midpoints of $\overline{BC}$, $\overline{DE}$, $\overline{FA}$ are the vertices of an equilateral triangle.

B-2. Let $0\leqq p\leqq 1$ and $0\leqq r\leqq 1$ and consider the identities

(a) $(px+(1-p)y)^2=Ax^2+Bxy+Cy^2$,

(b) $(px+(1-p)y)(rx+(1-r)y)=\alpha x^2+\beta xy+\gamma y^2$.

Show that (with respect to p and r)

(a) $\max\{A, B, C\}\geqq 4/9$,

(b) $\max\{\alpha, \beta, \gamma\}\geqq 4/9$.

B-3. If f and g are continuous and periodic functions with period 1 on the real line, then $\lim_{n\to\infty} \int_0^1 f(x)\, g(nx)\, dx = (\int_0^1 f(x)dx)(\int_0^1 g(x)dx)$.

B-4. (a) A certain locker room contains n lockers numbered $1, 2, 3, \cdots, n$ and all are originally locked. An attendant performs a sequence of operations $T_1, T_2, \cdots, T_n$ whereby with the operation T_k, $1 \leqq k \leqq n$, the condition of being locked or unlocked is changed for all those lockers and only those lockers whose numbers are multiples of k. After all the n operations have been performed it is observed that all lockers whose numbers are perfect squares (and only those lockers) are now open or unlocked. Prove this mathematically.

(b) Investigate in a meaningful mathematical way a procedure or set of operations similar to those above which will produce the set of cubes, or the set of numbers of the form $2m^2$, or the set of numbers of the form m^2+1, or some nontrivial similar set of your own selection.

B-5. Show that the sum of the first n terms in the binomial expansion of $(2-1)^{-n}$ is $\frac{1}{2}$, where n is a positive integer.

B-6. Let f be a real-valued function having partial derivatives and which is defined for $x^2+y^2 \leqq 1$ and is such that $|f(x, y)| \leqq 1$. Show that there exists a point (x_0, y_0) in the interior of the unit circle such that

$$\left(\frac{\partial f}{\partial x}(x_0, y_0)\right)^2 + \left(\frac{\partial f}{\partial y}(x_0, y_0)\right)^2 \leqq 16.$$

THE TWENTY-NINTH WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

December 7, 1968

A-1. Prove

$$\frac{22}{7} - \pi = \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx.$$

A-2. Given integers a, b, e, c, d, and f with $ad \neq bc$, and given a real number $\epsilon > 0$, show that there exist rational numbers r and s for which

$$0 < |ra + sb - e| < \epsilon,$$
$$0 < |rc + sd - f| < \epsilon.$$

A-3. Prove that a list can be made of all the subsets of a finite set in such a way that (i) the empty set is first in the list, (ii) each subset occurs exactly once, (iii) each subset in the list is obtained either by adding one element to the preceding subset or by deleting one element of the preceding subset.

A-4. Given n points on the sphere $\{(x, y, z): x^2+y^2+z^2=1\}$, demonstrate that the sum of the squares of the distances between them does not exceed n^2.

A-5. Let V be the collection of all quadratic polynomials P with real coefficients such that $|P(x)| \leqq 1$ for all x on the closed interval $[0, 1]$. Determine

$$\sup\{|P'(0)| : P \in V\}.$$

A-6. Determine all polynomials of the form $\sum_0^n a_i x^{n-i}$ with $a_i = \pm 1$ $(0 \leqq i \leqq n,\ 1 \leqq n < \infty)$ such that each has only real zeros.

B-1. The temperatures in Chicago and Detroit are $x°$ and $y°$, respectively. These temperatures are not assumed to be independent; namely, we are given:
(i) $P(x° = 70°)$, the probability that the temperature in Chicago is 70°,
(ii) $P(y° = 70°)$, and
(iii) $P(\max(x°, y°) = 70°)$.
Determine $P(\min(x°, y°) = 70°)$.

B-2. A is a subset of a finite group G (with group operation called multiplication), and A contains more than one half of the elements of G. Prove that each element of G is the product of two elements of A.

B-3. Assume that a 60° angle cannot be trisected with ruler and compass alone. Prove that if n is a positive multiple of 3, then no angle of $360/n$ degrees can be trisected with ruler and compass alone.

B-4. Show that if f is real-valued and continuous on $(-\infty, \infty)$ and $\int_{-\infty}^{\infty} f(x)dx$ exists, then

$$\int_{-\infty}^{\infty} f\left(x - \frac{1}{x}\right) dx = \int_{-\infty}^{\infty} f(x)dx.$$

B-5. Let p be a prime number. Let J be the set of all 2×2 matrices $\binom{ab}{cd}$ whose entries are chosen from $\{0, 1, 2, \cdots, p-1\}$ and satisfy the conditions $a+d \equiv 1 \pmod p$, $ad - bc \equiv 0 \pmod p$.

Determine how many members J has.

B-6. A set of real numbers is called compact if it is closed and bounded. Show that there does not exist a sequence $\{K_n\}_{n=0}^{\infty}$ of compact sets of rational numbers such that each compact set of rationals is contained in at least one K_n.

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Gerald L. Alexanderson, Leonard F. Klosinski and Loren C. Larson
Excerpt
More information

THE THIRTIETH WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

December 6, 1969

A-1. Let $f(x, y)$ be a polynomial with real coefficients in the real variables x and y defined over the entire x-y plane. What are the possibilities for the range of $f(x, y)$?

A-2. Let D_n be the determinant of order n of which the element in the ith row and the jth column is the absolute value of the difference of i and j. Show that D_n is equal to

$$(-1)^{n-1}(n-1)2^{n-2}.$$

A-3. Let P be a non-self-intersecting closed polygon with n sides. Let its vertices be P_1, $P_2, \cdots, P_n$. Let m other points, $Q_1, Q_2, \cdots, Q_m$ interior to P be given. Let the figure be triangulated. This means that certain pairs of the $(n+m)$ points $P_1, \cdots, Q_m$ are connected by line segments such that (i) the resulting figure consists exclusively of a set T of triangles, (ii) if two different triangles in T have more than a vertex in common then they have exactly a side in common, and (iii) the set of vertices of the triangles in T is precisely the set of $(n+m)$ points $P_1, \cdots, Q_m$. How many triangles in T?

A-4. Show that

$$\int_0^1 x^x dx = \sum_{n=1}^{\infty} (-1)^{n+1} n^{-n}.$$

(The integrand is taken to be 1 at $x=0$.)

A-5. Let $u(t)$ be a continuous function in the system of differential equations

$$\frac{dx}{dt} = -2y + u(t), \qquad \frac{dy}{dt} = -2x + u(t).$$

Show that, regardless of the choice of $u(t)$, the solution of the system which satisfies $x=x_0$, $y=y_0$ at $t=0$ will never pass through (0, 0) unless $x_0=y_0$. When $x_0=y_0$, show that, for any positive value t_0 of t, it is possible to choose $u(t)$ so the solution is at (0, 0) when $t=t_0$.

A-6. Let a sequence $\{x_n\}$ be given, and let $y_n=x_{n-1}+2x_n$, $n=2, 3, 4, \cdots$. Suppose that the sequence $\{y_n\}$ converges. Prove that the sequence $\{x_n\}$ also converges.

B-1. Let n be a positive integer such that $n+1$ is divisible by 24. Prove that the sum of all the divisors of n is divisible by 24.

B-2. Show that a finite group can not be the union of two of its proper subgroups. Does the statement remain true if "two" is replaced by "three"?

B-3. The terms of a sequence T_n satisfy

$$T_n T_{n+1} = n \qquad (n = 1, 2, 3, \cdots) \text{ and } \lim_{n\to\infty} \frac{T_n}{T_{n+1}} = 1.$$

Show that $\pi T_1^2 = 2$.

B-4. Show that any curve of unit length can be covered by a closed rectangle of area 1/4.