Proofs That Really Count
The Art of Combinatorial Proof
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Proofs That Really Count
The Art of Combinatorial Proof

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Dedicated to our families—the people who count the most in our lives.

To Deena, Laurel, and Ariel. A.T.B.

To the bad boys of Loleta. J.J.Q.
Foreword

Every proof in this book is ultimately reduced to a counting problem—typically enumerated in two different ways. Counting leads to beautiful, often elementary, and very concrete proofs. While not necessarily the simplest approach, it offers another method to gain understanding of mathematical truths. To a combinatorialist, this kind of proof is the only right one. We offer Proofs That Really Count as the counting equivalent of the visual approach taken by Roger Nelsen in Proofs Without Words I & II [37, 38].

Why count?

As human beings we learn to count from a very early age. A typical 2 year old will proudly count to 10 for the coos and applause of adoring parents. Though many adults readily claim ineptitude in mathematics, no one ever owns up to an inability to count. Counting is one of our first tools, and it is time to appreciate its full mathematical power. The physicist Ernst Mach even went so far as to say, “There is no problem in all mathematics that cannot be solved by direct counting” [36].

Combinatorial proofs can be particularly powerful. To this day, I (A.T.B.) remember my first exposure to combinatorial proof when I was a freshman in college. My professor proved the Binomial Theorem

\[(x + y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}\]

by writing

\[(x + y)^n = (x + y)(x + y) \cdots (x + y)\]

n times

and asking “In how many ways can we create an \(x^k y^{n-k}\) term?” Sudden clarity ensued. The theorem made perfect sense. Yes, I had seen proofs of the Binomial Theorem before, but they had seemed awkward and I wondered how anyone in his or her right mind would create such a result. But now it seemed very natural. It became a result I would never forget.

What to count?

We have selected our favorite identities using numbers that arise frequently in mathematics (binomial coefficients, Fibonacci numbers, Stirling numbers, etc.) and have chosen elegant counting proofs. In a typical identity, we pose a counting question, and then answer it in
two different ways. One answer is the left side of the identity; the other answer is the right side. Since both answers solve the same counting question, they must be equal. Thus the identity can be viewed as a counting problem to be tackled from two different angles.

We use the identity

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

to illustrate a proof structure found throughout this book. There is no need to use the formula \( \frac{n!}{k!(n-k)!} \) for \( \binom{n}{k} \). Instead, we interpret \( \binom{n}{k} \) as the number of \( k \)-element subsets of an \( n \)-element set, or more colorfully, as the number of ways to select a committee of \( k \) students from a class of \( n \) students.

**Question:** From a class of \( n \) students, how many ways can we create a committee?

**Answer 1:** The number of committees with 0 students is \( \binom{n}{0} \). The number of committees with 1 student is \( \binom{n}{1} \). In general, the number of committees with exactly \( k \) students is \( \binom{n}{k} \). Hence the total number of committees is \( \sum_{k=0}^{n} \binom{n}{k} \).

**Answer 2:** To create a committee of arbitrary size, we decide, student by student whether or not they will be on the committee. Since each of the \( n \) students is either “on” or “off” the committee, there are 2 possibilities for each student and thus \( 2^n \) ways to create a committee.

Since our logic is impeccable in both answers, they must be equal, and the identity follows.

Another useful proof technique is to interpret the left side of an identity as the size of a set, the right side of the identity as the size of a different set, and then find a one-to-one correspondence between the two sets. We illustrate this proof structure with the identity

$$\sum_{k \geq 0} \binom{n}{2k} = \sum_{k \geq 0} \binom{n}{2k+1}$$

for \( n > 0 \).

Both sums are finite since \( \binom{n}{i} = 0 \) whenever \( i > n \). Here it is easy to see what both sides count. The challenge is to find the correspondence between them.

**Set 1:** The committees with an even number of members formed from a class of \( n \) students. This set has size \( \sum_{k \geq 0} \binom{n}{2k} \).

**Set 2:** The committees with an odd number of members formed from a class of \( n \) students. This set has size \( \sum_{k \geq 0} \binom{n}{2k+1} \).

**Correspondence:** Suppose one of the students in the class is named Waldo. Any committee with an even number of members can be turned into a committee with an odd number of members by asking “Where’s Waldo?” If Waldo is on the committee, then remove him. If Waldo is not on the committee, then add him. Either way, the parity of the committee has changed from even to odd.

Since the process of “removing or adding Waldo” is completely reversible, we have a one-to-one correspondence between these sets. Thus both sets must have the same size, and the identity follows.
FOREWORD

Often we shall prove an identity more than one way, if we think a second proof can bring new insight to the problem. For instance, the last identity can be handled by counting the number of even subsets directly. See Identity 129 and the subsequent discussion.

What can you expect when reading this book? Chapter 1 introduces a combinatorial interpretation of Fibonacci numbers as square and domino tilings, which serves as the foundation for Chapters 2–4. We begin here because Fibonacci numbers are intrinsically interesting and their interpretation as combinatorial objects will come as a delightful surprise to many readers. As with all the chapters, this one begins with elementary identities and simple arguments that help the reader to gain a familiarity with the concepts before proceeding to more complex material. Expanding on the Fibonacci tilings will enable us to explore identities involving generalized Fibonacci numbers including Lucas numbers (Chapter 2), arbitrary linear recurrences (Chapter 3), and continued fractions (Chapter 4).

Chapter 5 approaches the traditional combinatorial subject of binomial coefficients. Counting sets with and without repetition leads to identities involving binomial coefficients. Chapter 6 looks at binomial identities with alternating signs. By finding correspondences between sets with even numbers of elements and sets with odd numbers of elements, we avoid using the familiar method of overcounting and undercounting provided by the Principle of Inclusion-Exclusion.

Harmonic numbers, like continued fractions, are not integral—so a combinatorial explanation requires investigating the numerator and denominator of a particular representation. Harmonic numbers are connected to Stirling numbers of the first kind. Chapter 7 investigates and exploits this connection in addition to identities involving Stirling numbers of the second kind.

Chapter 8 considers more classical results from arithmetic, number theory, and algebra including the sum of consecutive integers, the sum of consecutive squares, sum of consecutive cubes, Fermat’s Little Theorem, Wilson’s Theorem, and a partial converse to Lagrange’s Theorem.

In Chapter 9, we tackle even more complex Fibonacci and binomial identities. These identities require ingenious arguments, the introduction of colored tiles, or probabilistic models. They are perhaps the most challenging in the book, but well worth your time.

Occasionally, we digress from identities to prove fun applications. Look for a divisibility proof on Fibonacci numbers in Chapter 1, a magic trick in Chapter 2, a shortcut to calculate the parity of binomial coefficients in Chapter 5 and generalizations to congruences modulo arbitrary primes in Chapter 8.

Each chapter, except the last, includes a set of exercises for the enthusiastic reader to try his or her own counting skills. Most chapters contain a list of identities for which combinatorial proofs are still being sought. Hints and references for the exercises and a complete listing of all the identities can be found in the appendices at the end of the book.

Our hope is that each chapter can stand independently, so that you can read in a nonlinear fashion if desired.

Who should count?

The short answer to this question is “Everybody counts!” We hope this book can be enjoyed by readers without special training in mathematics. Most of the proofs in this book can be appreciated by students at the high school level. On the other hand, teachers may find this book to be a valuable resource for classes that emphasize proof writing and creative problem solving techniques. We do not consider this book to be a complete
survey of combinatorial proofs. Rather, it is a beginning. After reading it, you will never view quantities like Fibonacci numbers and continued fractions the same way again. Our hope is that an identity like

\[ f_{2n+1} = \sum_{i=0}^{n} \sum_{j=0}^{n} \binom{n-i}{j} \binom{n-j}{i} \]

for Fibonacci numbers should give you the feeling that something is being counted and the desire to count it. Finally, we hope this book will serve as an inspiration for mathematicians who wish to discover combinatorial explanations for old identities or discover new ones. We invite you, our readers, to share your favorite combinatorial proofs with us for (possible) future editions.

After all, we hope all of our efforts in writing this book will count for something.

Who counts?

We are pleased to acknowledge the many people who made this book possible—either directly or indirectly.

Those who came before us are responsible for the rise in popularity of combinatorial proof. Books whose importance cannot be overlooked are Constructive Combinatorics by Dennis Stanton and Dennis White, Enumerative Combinatorics Volumes 1 & 2 by Richard Stanley, Combinatorial Enumeration by Ian Goulden and David Jackson, and Concrete Mathematics by Ron Graham, Don Knuth & Oren Patashnik. In addition to these mathematicians, others whose works continue to inspire us include George E. Andrews, David Bressoud, Richard Brualdi, Leonard Carlitz, Ira Gessel, Adriano Garsia, Ralph Grimaldi, Richard Guy, Stephen Milne, Jim Propp, Marta Sved, Herbert Wilf, and Doron Zeilberger.

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Contents

Foreword ix

1 Fibonacci Identities 1
   1.1 Combinatorial Interpretation of Fibonacci Numbers 1
   1.2 Identities 2
   1.3 A Fun Application 11
   1.4 Notes 12
   1.5 Exercises 13

2 Gibonacci and Lucas Identities 17
   2.1 Combinatorial Interpretation of Lucas Numbers 17
   2.2 Lucas Identities 18
   2.3 Combinatorial Interpretation of Gibonacci Numbers 23
   2.4 Gibonacci Identities 23
   2.5 Notes 32
   2.6 Exercises 32

3 Linear Recurrences 35
   3.1 Combinatorial Interpretations of Linear Recurrences 36
   3.2 Identities for Second-Order Recurrences 38
   3.3 Identities for Third-Order Recurrences 40
   3.4 Identities for 4th Order Recurrences 43
   3.5 Get Real! Arbitrary Weights and Initial Conditions 44
   3.6 Notes 45
   3.7 Exercises 45

4 Continued Fractions 49
   4.1 Combinatorial Interpretation of Continued Fractions 49
   4.2 Identities 52
   4.3 Nonsimple Continued Fractions 58
   4.4 Get Real Again? 59
   4.5 Notes 59
   4.6 Exercises 60

xiii
5 Binomial Identities 63
  5.1 Combinatorial Interpretations of Binomial Coefficients ............. 63
  5.2 Elementary Identities ........................................... 64
  5.3 More Binomial Coefficient Identities ............................ 68
  5.4 Multichoosing ...................................................... 70
  5.5 Odd Numbers in Pascal’s Triangle .................................. 75
  5.6 Notes ................................................................ 77
  5.7 Exercises ............................................................. 78

6 Alternating Sign Binomial Identities 81
  6.1 Parity Arguments and Inclusion-Exclusion ........................... 81
  6.2 Alternating Binomial Coefficient Identities ........................ 84
  6.3 Notes ................................................................ 89
  6.4 Exercises ............................................................. 89

7 Harmonic and Stirling Number Identities 91
  7.1 Harmonic Numbers and Permutations ................................. 91
  7.2 Stirling Numbers of the First Kind ................................... 93
  7.3 Combinatorial Interpretation of Harmonic Numbers ................ 97
  7.4 Recounting Harmonic Identities ...................................... 98
  7.5 Stirling Numbers of the Second Kind ................................ 103
  7.6 Notes ................................................................ 106
  7.7 Exercises ............................................................. 106

8 Number Theory 109
  8.1 Arithmetic Identities ................................................... 109
  8.2 Algebra and Number Theory .......................................... 114
  8.3 GCDs Revisited .......................................................... 118
  8.4 Lucas’ Theorem .......................................................... 120
  8.5 Notes ................................................................ 123
  8.6 Exercises ............................................................. 123

9 Advanced Fibonacci & Lucas Identities 125
  9.1 More Fibonacci and Lucas Identities ................................ 125
  9.2 Colorful Identities ...................................................... 130
  9.3 Some “Random” Identities and the Golden Ratio .................. 136
  9.4 Fibonacci and Lucas Polynomials .................................... 141
  9.5 Negative Numbers ..................................................... 143
  9.6 Open Problems and Vajda Data ...................................... 143

Some Hints and Solutions for Chapter Exercises 147

Appendix of Combinatorial Theorems 171

Appendix of Identities 173

Bibliography 187

Index 191

About the Authors 194