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A Tour through Mathematical Logic

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A Tour through Mathematical Logic

Robert S. Wolf
California Polytechnic State University

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Preface

This book is supposed to be about "mathematical logic," so it seems appropriate to begin with a definition of that phrase. On the surface, this is an easy task: we can say that mathematical logic is the branch of mathematics that studies the logic and the methods of deduction used in mathematics. Chapter 1 is an introduction to mathematical logic in this narrow sense.

A bit of reflection should convince you that this is not a typical branch of mathematics. Most branches of mathematics study mathematical "objects" or "structures": numbers, triangles, functions of a real variable, groups, topological spaces, etc. But mathematical logic studies something very different—essentially, mathematical reasoning.

A couple of points come to mind on the basis of this characterization. One is that mathematical logic could be among the most abstract branches of mathematics, since it is particularly difficult to give a concrete description of its subject material. (On the other hand, mathematical logic also studies the symbolic language used in mathematics, which is a concrete object.) Another is that there is something circular, perhaps even paradoxical, in the nature of this field. For its subject matter is mathematical reasoning, but since it is a branch of mathematics its method of inquiry must also consist of mathematical reasoning. The term "metamathematics" was introduced by David Hilbert to describe this unusual situation: a discipline that is mathematical in nature but at the same time beyond or above ordinary mathematics because it treats

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mathematics as an object of study. (Another branch of mathematics with a somewhat similar status is category theory.)

However, there are compelling reasons to broaden what we mean by mathematical logic. It turns out to be of limited value to study mathematical reasoning without also addressing the nature of mathematical objects. After all, reasoning is usually *about* something, and if it is, understanding what the reasoning is about may be nearly essential to analyzing the reasoning itself. One of the great achievements of modern mathematics has been the unification of the many types of objects studied in mathematics. This unification began in the seventeenth century, when René Descartes and Pierre de Fermat showed how geometric objects (points, lines, circles, etc.) could be represented numerically or algebraically, and culminated in the twentieth century with the sophisticated use of set theory to represent all the objects normally considered in mathematics. In common parlance (including the title of this book), the term "mathematical logic" is used as a synonym for "foundations of mathematics," which includes both mathematical logic in the stricter sense defined above, and the study of the fundamental types of mathematical objects and the relationships among them.

Accordingly, this book includes two chapters on set theory, one on the basics of the subject and the other on the explosion of methods and results that have been discovered since the 1960s. It also includes a chapter on model theory, which uses set theory to show that the theorem-proving power of the usual methods of deduction in mathematics corresponds perfectly to what must be true in actual mathematical structures. The remaining chapters describe several other important branches of the foundations of mathematics. Chapter 3 is about the important concepts of "effective procedures" and "computable functions," and Chapter 4 describes Kurt Gödel's (and others') famous incompleteness theorems. Chapter 7 is on the powerful tool known as nonstandard analysis, which uses model theory to justify the use of infinitesimals in mathematical reasoning. The final chapter is about various "constructivist" movements that have attempted to convince mathematicians to give "concrete" reasoning more legitimacy than "abstract" reasoning.

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The word "tour" in the title of this book also deserves some explanation. For one thing, I chose this word to emphasize that it is not a textbook in the strict sense. To be sure, it has many of the features of a textbook, including exercises. But it is less structured, more free-flowing, than a standard text. It also lacks many of the details and proofs that one normally expects in a mathematics text. However, in almost all such cases there are references to more detailed treatments and the omitted proofs. Therefore, this book is actually quite suitable for use as a text at the university level (undergraduate or graduate), provided that the instructor is willing to provide supplementary material from time to time.

The most obvious advantage of this omission of detail is that this monograph is able to cover a lot more material than if it were a standard textbook of the same size. This de-emphasis on detail is also intended to help the reader concentrate on the big picture, the essential ideas of the subject, without getting bogged down in minutiae. Could this book have been titled "A Survey of Mathematical Logic"? Perhaps it could have, but my choice of the word "tour" was deliberate. A survey sounds like a rather dry activity, carried out by technicians with instruments. Tours, on the other hand, are what people take on their vacations. They are intended to be fun. On a tour of Europe, you might learn a lot about the cathedrals of Germany or Impressionist art, but you wouldn't expect to learn about them as thoroughly as if you took a course in the subject. The goal of this book is similar: to provide an introduction to the foundations of mathematics that is substantial and stimulating, and at the same time a pleasure to read.

Here are some suggestions for how to proceed on the tour that this book provides. It is self-contained enough that a reader with no previous knowledge of logic and set theory (but at least a year of mathematics beyond calculus) should be able to read it on his or her own, without referring to outside sources and without getting seriously stuck. This is especially true of the first two chapters, which (except for Section 1.7) consist of basic material that is prerequisite to the later chapters. Before proceeding to the later chapters, it is highly advisable to read and understand at least Sections 1.1 through 1.5 and 2.1 through 2.3.

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Several appendices covering some important background material are provided for the reader with limited experience in higher mathematics. The main topics included in these appendices, other than those from foundations, are ordering relations and basic concepts from abstract algebra (groups, rings, and fields). Some topics that are not included are basic concepts from linear algebra (e.g., vector spaces), analysis, and topology. These topics are occasionally mentioned in the book, but the reader who is unfamiliar with them should not be significantly hindered.

The remaining chapters are distinctly more advanced than the first two. Attempting to digest them without first understanding the basics of logic and set theory could prove frustrating, but the savvy reader may be able to read one or more of these later chapters, needing to refer only occasionally to particular material in Chapters 1 and 2. Most of these later chapters depend only slightly on each other. So, for instance, someone who decides to read Chapters 1, 2, and 5 will not be significantly hindered by having skipped Chapters 3 and 4. The most significant exceptions to this are that Chapter 4 is based essentially on Chapter 3, and Chapters 6 and 7 use many concepts from Chapter 5.

Of course, even in the introductory chapters, if you want to learn the material thoroughly, then you must be prepared to reread certain parts, do the exercises, and perhaps fill in some details from outside sources. A book can be made pleasant to read, but it cannot turn abstract mathematics into a walk in the park. As Euclid supposedly said to King Ptolemy, "There is no royal road to geometry."

This book includes about a dozen biographies, in boxes, of people who have made important contributions to mathematical logic and foundations. In order to fend off any bewilderment about whom I chose for these and why, let me clarify two points. First, I decided not to include biographies of any living mathematicians. Second, my choices were motivated only partly by the importance, quality, quantity, brilliance, etc., of the person's contributions. My other main criterion was to what extent the person's life was noteworthy or unusual, perhaps even heroic or tragic. So I have omitted the biographies of some true "giants" in the field whose lives were relatively straightforward.

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I sincerely hope that you will find this "tour" interesting, informative, and enjoyable. I welcome any type of feedback, including questions, corrections, and criticisms, sent to rswolf@calpoly.edu.

Acknowledgments

One of the few pleasurable activities that normally occurs during the frantic process of finalizing a book is thanking the people who helped the author(s) along the way. In the present situation, an unusually large number of people have aided me in one way or another, and I am genuinely grateful to all of them.

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Space does not allow me to thank all the mathematicians and mathematics students who have read part or all of this manuscript and provided valuable feedback. So, at the risk of omitting some deserving ones (and with apologies in advance, if that occurs), I want to thank the following people who have kindly donated a substantial amount of time to provide comments and guidance: Michael Beeson, Jim Henle, Robert Solovay, Charlie Steinhorn, Todor Todorov, and Dan Velleman.

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Finally, I want to thank my friends and family for their patience and encouragement during these several years of distraction and A.R.-ness on my part. Special kudos go to my wife, Laurie, for suppressing her impulse to laugh out loud when the number of times I said, "I've just finished my book!" hit double digits.

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