CHAPTER 1
Predicate Logic

1.1 Introduction

Why should students of mathematics want to know something about predicate logic? Here is one answer: predicate logic helps one understand the fine points of mathematical language, including the ambiguities that arise from the use of natural language (English, French, etc.) in mathematics. For research mathematicians, these ambiguities are usually easy to spot, and their intuition will normally prevent any errors. Yet, occasionally, such ambiguities can be very subtle and present real pitfalls. For instructors of mathematics, awareness of this type of ambiguity is indispensable for analyzing students’ mistakes and misconceptions.

Let’s illustrate this point with a rather fanciful example: imagine that you are an eminent mathematician with an international reputation. Your life is wonderful in every way except that, after many years, your marriage has become stale. Reluctantly, you decide to have a fling, hoping that a brief one will be enough and your marriage will survive. You approach your friend Lee, whose marriage is also stale, and the two of you agree to have a tryst. (For the record, I would never do such a thing. Remember, it’s you I’m talking about. If you wouldn’t do such a thing either, then I must be thinking of a different reader.)
To be on the safe side, you and Lee decide to go to another state for your tryst. You drive there and go to a nondescript motel. To your surprise, the desk clerk asks, “Are you folks married? We have to ask that, and you’re required by state law to answer truthfully.” Without batting an eyelash, you reply, “Of course we are.” You get the room and have your little fling, but the next morning you are arrested—not for adultery (since nothing can be proved about that), but for perjury!

From your jail cell, you plan your clever defense: you will simply say that you didn’t lie at all. You and Lee are married, but not to each other! Your answer was literally quite correct, so no perjury has occurred.

Will your defense succeed? I doubt it, but that’s not what interests us anyway. Among the many types of ambiguities in the English language, this one is of a particularly interesting sort. Note that the statement “I am married” is not ambiguous. The statement (or predicate, since it involves a variable) “x is married” means that x is married to someone. But the statement “x and y are married” somehow becomes ambiguous. It would normally mean that x and y are married to each other, but it could mean that x and y are, individually, married persons. To analyze the source of this ambiguity, one should realize that the binary (two-variable) predicate “x and y are married to each other” is really the basic or atomic predicate here. If we use $M(x, y)$ to denote this sentence, then the unary (one-variable) predicate “x is married” actually means “There is a person y such that $M(x, y)$.” It is not atomic because it includes a quantifier (“there is”), which is tacit in its usual English rendition. Thus the less common meaning of “x and y are married” is in a sense simpler than the usual meaning (because it’s a statement about the individuals x and y separately), but in another sense more complicated (because it involves two quantifiers).

We are done with this example, but one could concoct other interesting questions. How would you normally interpret “Fred and Jane and Bill and Mary are married”? How about “Fred, Jane, Bill and Mary are married”? How about “Fred, Jane, and Bill are married”? Of course, we usually rely on context to interpret such statements, but sometimes there is little or no context to rely on.
Another example: Suppose you were told that “two sisters entered the room.” Normally, this would mean that two women who are each other’s sisters entered the room (binary predicate of sisterhood). It would be quite artificial to make this statement to express that the women both had siblings. But suppose you were told that “two sisters entered the convent.” Suddenly a different meaning presents itself, in which sisterhood is a unary predicate, completely unrelated to the usual binary predicate.

As a final nonmathematical example, I recently found myself at lunch in a restaurant with two married couples. When the waitress asked if we wanted a single check or separate ones, I spontaneously responded, “Those two are together, and so are those two, but I’m not very together.”

Does this sort of ambiguity occur in mathematics? Yes it does, frequently and in many guises. An example that I like to use as a pedagogical tool is “Let C be a collection of nonempty, disjoint sets.” The professional mathematician has no problem understanding this sentence. But the beginning student can easily forget that nonemptiness is a property of individual sets, a unary predicate, while disjointness is a property of pairs of sets, a binary predicate. Sometimes, the word “pairwise” is inserted before “disjoint” to clarify this difference. Also, note that the statement that two sets are disjoint is symmetric, so it is essentially a predicate on unordered pairs of (distinct) sets rather than ordered pairs of sets. If this were not the case, the notion of a collection of disjoint sets would make no sense (but the notion of a sequence of disjoint sets could be meaningful).

A particularly complex example of this sort is the statement that a set $B$ of vectors in an inner product space is an orthonormal basis. One part of this statement is that each member of $B$ is a unit vector (a unary predicate). Another part is that the vectors are orthogonal (a binary predicate, about unordered pairs of distinct vectors). Finally, the property of being a basis is essentially a statement about the whole set $B$, as opposed to a statement about all individuals in $B$, or all pairs in $B$, etc. But a bit more can be said here: the linear independence of $B$ is equivalent to the linear independence of every finite subset of $B$. So,
4

Predicate Logic

if one desires, one may view linear independence as a predicate that is initially defined for finite sets of vectors. But the statement that $B$ spans the space is not equivalent to saying that some (or every) finite subset of $B$ spans the space. It has the more complex form, “For every vector $v$, there is a finite subset of $B$ such that $v$ is a linear combination of the vectors in that finite subset.”

Of course, mathematical logic is about a lot more than analyzing ambiguities. The study of logic and the foundations of mathematics has led to many powerful and versatile methods, and many deep and beautiful results. Some of these methods and results are relevant within foundations only, but many of them are useful and important in other branches of mathematics and in fields outside of mathematics. The goal of this book is to show you some of the power, depth, and beauty of this subject.

A very brief history of mathematical logic

The basic principles of logic and its use in mathematics were well understood by the philosophers and mathematicians of the classical Greek era, and Aristotle provided the first systematic written treatment of logic (propositional logic, primarily). Aristotle’s description of logic was carried out in the context of natural language, and so it is considered informal rather than symbolic. It was not until two thousand years later (almost exactly!) that the seeds of modern mathematical logic were sown, with Aristotle’s work as a basis. In the late 1600s Gottfried Leibniz, one of the co-inventors of calculus, expressed and worked toward the goal of creating a symbolic language of reasoning that could be used to solve all well-defined problems, not just in mathematics but in science and other disciplines as well. We have just seen several examples of the ambiguities that are inherent in English and other natural languages. Leibniz understood clearly that such ambiguities could be problematic in a discipline as exacting as mathematics.

Almost two more centuries passed before Augustus De Morgan and George Boole extended Leibniz’s work and began to create modern symbolic logic. De Morgan is best remembered for the two laws of
propositional logic that bear his name, while Boolean algebra found many uses in the twentieth century, notably the essential design principles of computer circuitry. Finally, predicate logic as we know it today was created in the latter part of the 1800s by Gottlob Frege and, a few years later but independently, by Giuseppe Peano. Frege gave the first rigorous treatment of the logic of quantifiers, while Peano invented many of the symbols currently used in propositional logic and set theory. Both of them applied their work to create formal theories of arithmetic. Of these, it is Peano’s theory that has since found wide acceptance (see Example 16 in Section 1.5), but Frege’s work had more influence on the development of modern mathematical logic.

The contradictions that were discovered in set theory at the end of the nineteenth century (to be discussed in Chapter 2) heightened awareness of the potential dangers of informal reasoning in mathematics. As a result, one of the goals of those wanting to free mathematics from such contradictions was to reduce the importance of natural language in mathematics. One monumental effort toward this end was the logicist program of Bertrand Russell and Alfred North Whitehead, which extended the work of Peano and Frege. Their 2000-page Principia Mathematica [RW], which took a decade to write, attempted to formulate all of mathematics in terms of logic. They hoped to make logic not just a tool for doing mathematics, but also the underlying subject matter on which all mathematical concepts would be based. They did not quite succeed in this, but they provided much of the groundwork for the formalization of mathematics within set theory. (So mathematics can’t be based solely on logic, but it can be based on logic and set theory. In a sense, this is what Russell and Whitehead were trying to do, since their idea of logic included some set-theoretic concepts.)

In the wake of these achievements of Peano, Frege, Russell, and Whitehead, another ambitious plan was proposed by Hilbert. The goals of his formalist program were to put all of mathematics into a completely symbolic, axiomatic framework, and then to use metamathematics (mathematics applied to mathematics), also called proof theory, to show that mathematics is free from contradictions. Earlier, in the 1890s, Hilbert had published his Foundations of Geometry, con-
Bertrand Russell (1872–1970) was not primarily a mathematician but continued in the ancient tradition of philosophers who have made important contributions to the foundations of mathematics. He was born into a wealthy liberal family, orphaned at age four, and then raised by his grandmother, who had him tutored privately.

Russell made two great contributions to modern set theory and logic. One was his co-discovery of the inconsistency of naïve set theory, discussed in Section 2.2. The other was the monumental *Principia Mathematica* [RW], written with his former professor, Alfred North Whitehead. As a philosopher, Russell was one of the main founders of the modern analytic school. He was a prolific writer and wrote many books intended for the general public, notably the best-selling *A History of Western Philosophy* (1950), for which he won the Nobel Prize in Literature.

Outside of academia, Russell is best known for his political and social activism. During World War I, his pacifism led to his dismissal from Trinity College and a six-month prison sentence. Half a century later, he vehemently opposed nuclear weapons, racial segregation, and the U. S. involvement in Vietnam. He advocated sexual freedom and trial marriage as early as the 1930s, causing a court of law to nullify a faculty position that had been offered to him by the City College of New York in 1940.

By the way, the web site

http://www-history.mcs.st-andrews.ac.uk/history/index.html

run by the University of St. Andrews is an excellent source for biographical information about mathematicians.
Propositional logic

considered to be the first version of Euclidean geometry that was truly axiomatic, in the sense that there were no hidden appeals to spatial intuition. Referring to this work, Hilbert made the point that his proofs should stay completely correct even if the words “point,” “line,” and “plane” were replaced throughout by “table,” “chair,” and “beer mug.”

Because of the famous negative results obtained by Kurt Gödel, there is a tendency to think of the formalist program as a failure. But in fact two major parts of Hilbert’s program—the translation of mathematical language and mathematical proofs into a purely formal, symbolic format—succeeded extremely well.

Chapter 4 will provide a more detailed discussion of Hilbert’s formalist program and how Gödel’s work affected it. An excellent source for the history of mathematics and its foundations is [Bur]. See [Hei] or [BP] for important writings in foundations since 1879, in their original form.

In this chapter we will outline the key concepts of mathematical logic, the logic that underlies the practice of mathematics. Readers who want a more thorough introduction to this material at a fairly elementary level may want to consult [Wolf], [Men], [Ross], or [End]. A more advanced treatment is given in [Sho].

1.2 Propositional logic

This section and the next will cover the most elementary ideas of mathematical logic. Many people who have studied mathematics at the post-calculus level should be familiar with this material, either through a specific course (in logic, or perhaps an “introduction to higher mathematics” course) or through general experience. Readers in this category are encouraged to skip or skim these sections.

Informal treatments of propositional logic (also known as sentential logic or the propositional calculus) usually begin by defining a proposition to be simply “a declarative sentence that is either true or false.” Frankly, this definition is rather imprecise. Certainly, questions and commands are not propositions. Statements such as “Snow is
Predicate Logic

white” and “2 + 2 = 5” are often given as examples of propositions, while “x + 3 = 7” is not a proposition because it is not true or false as it stands. Its truth or falsity depends on the value of x, and so it is called a **propositional function** or a **predicate**.

But there are all sorts of declarative sentences whose inclusion as propositions is debatable. These include statements about the future (“The Cubs will win the World Series before 2099”), certain statements about the past (“The continent of Atlantis sank into the ocean”), value judgments (“Life is good”), abstract mathematical statements (“The continuum hypothesis is true”), and statements in which a noun or a pronoun is not clearly specified and thus is similar to a mathematical variable (“Bill is not the president”). Indeed, one could even question the classic “Snow is white,” since in reality snow is not always white.

However, these philosophical problems are not germane to our mathematical treatment of logic. Let us use the word **statement** to mean any declarative sentence (including mathematical ones such as equations) that is true or false or could become true or false in the presence of additional information.

We will use the **propositional variables** P, Q, and R (possibly with subscripts) to stand for statements.

Propositional logic studies the meaning of combinations of statements created by using certain words called **connectives**. The most commonly used connectives, and the symbols used to abbreviate them, are: “and” (∨), “or” (∨), “not” (¬), “implies” or “if . . . then” (→), and “if and only if” (⇔), commonly abbreviated “iff.” The grammatical rules for the use of the connectives are simple: if P and Q are statements, then so are P ∨ Q, P ∨ Q, P → Q, and P ⇔ Q.

There is nothing sacred about this set of connectives, or the number five. One could use many more connectives (e.g., “neither . . . not,” “not both,” “unless”); or one could be very economical and express all these meanings with just two connectives (such as “and” and “not”) or even with just one connective. In other words, there is plenty of redundancy among the five standard connectives.

There is some useful terminology associated with the connectives. A statement of the form P ∧ Q is called the **conjunction** of the two con-
Propositional logic

juncts $P$ and $Q$. Similarly, $P \lor Q$ is the disjunction of the conjuncts $P$ and $Q$, $\sim P$ is called the negation of $P$, while $P \rightarrow Q$ is called a conditional statement or an implication. In $P \rightarrow Q$, $P$ is the hypothesis or the antecedent of the implication, while $Q$ is its conclusion or consequent. Finally, $P \leftrightarrow Q$ is a biconditional statement or an equivalence.

If a statement contains more than one connective, parentheses may be used to make its meaning clear. In ordinary mathematics the need for parentheses is often eliminated by an understood priority of operations. For example, in algebra multiplication is given higher priority than addition, so $a + b \cdot c$ means $a + (b \cdot c)$ as opposed to $(a + b) \cdot c$. Similarly, we give the connectives the following priorities, from highest to lowest:

$$\sim, \land, \lor, \rightarrow, \leftrightarrow$$.

So $P \rightarrow Q \land R$ would mean $P \rightarrow (Q \land R)$ rather than $(P \rightarrow Q) \land R$. Parentheses can be used to convey the other meaning. Similarly, $\sim P \rightarrow Q$, with no parentheses, would mean $(\sim P) \rightarrow Q$ rather than $\sim (P \rightarrow Q)$.

A statement is called atomic if it has no connectives or quantifiers, and compound otherwise. (Quantifiers are introduced in the next section.)

In classical or Aristotelian logic, which is almost universally accepted as the appropriate type of logic for mathematics, the meaning of the connectives is defined by truth functions, also called truth tables. These are literally functions whose inputs and outputs are the words "true" and "false" (or $T$ and $F$) instead of numbers. Specifically, $P \land Q$ is true if and only if both $P$ and $Q$ are true. $P \lor Q$ is true if and only if at least one of $P$ and $Q$ is true. $\sim P$ is true if and only if $P$ is false. $P \rightarrow Q$ is true unless $P$ is true and $Q$ is false. $P \leftrightarrow Q$ is true if and only if $P$ and $Q$ are both true or both false.

Two of these truth tables deserve some discussion. The truth table for $P \lor Q$ defines the so-called inclusive or, in which $P \lor Q$ is considered true when both $P$ and $Q$ are true. Thus, for instance, "$2 + 2 = 4$ or $5 > \pi$" is a true statement. One could also define the exclusive or, where "$P$ xor $Q$" is true if and only if exactly one of $P$ or $Q$ is true.
Predicate Logic

Every mathematics student needs to develop an understanding of the logical meaning of implication. Note that \( P \rightarrow Q \) is true in three of the four possible cases. In particular, it is automatically true if \( P \) is false. For example, "If 2 + 2 = 3, then Mars has seven moons" is true. In ordinary speech, a statement of the form "\( P \) implies \( Q \)" normally asserts that there is some causal relationship between \( P \) and \( Q \), that the truth of \( P \) somehow causes the truth of \( Q \). This is not required in mathematics or logic (or in sarcastic implications such as "If you can run a six minute mile, then I'm the Queen of England").

It is also worth noting that, among the four basic connectives that involve more than one substatement, implication is the only one that is not symmetrical in \( P \) and \( Q \). It is very important not to confuse an implication \( P \rightarrow Q \) with its converse \( Q \rightarrow P \).

A statement that is built up from simpler ones using connectives only is called a propositional combination or a Boolean combination of those statements. The truth functions of the connectives allow us to determine the truth function of any propositional combination of statements. For readability, truth functions are usually written for statements that are built up from propositional variables. It is easy to show that if a symbolic statement has \( n \) propositional variables, then the domain of its truth function consists of \( 2^n \) combinations of Ts and Fs.

**Example 1.** Here is a truth table for the symbolic statement

\[ (P \rightarrow Q) \leftrightarrow (R \land P). \]

<table>
<thead>
<tr>
<th>P</th>
<th>Q</th>
<th>R</th>
<th>P \rightarrow Q</th>
<th>R \land P</th>
<th>(P \rightarrow Q) \leftrightarrow (R \land P)</th>
</tr>
</thead>
<tbody>
<tr>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>T</td>
</tr>
<tr>
<td>T</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>T</td>
<td>F</td>
</tr>
<tr>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>T</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
<td>F</td>
</tr>
<tr>
<td>F</td>
<td>F</td>
<td>F</td>
<td>T</td>
<td>F</td>
<td>F</td>
</tr>
</tbody>
</table>