

Index

- reaction–diffusion equation
 - Dirichlet boundary conditions, 103
 - finite element method (FEM), **120**
 - Neumann boundary conditions, 108
 - periodic boundary conditions, 103
 - stochastic, 463
- adapted, 319
- advection equation, 98
- Allen–Cahn equation
 - deterministic, 114, 115
 - stochastic, 432, 468
- almost surely, 138
- analytic, 482
- antithetic samples, 176
- a priori (error estimate), 376, 388
- a.s., *see* almost surely
- backward Kolmogorov equation, 350
- Banach space, **1**, 2
- Berry–Esséen inequality, 161
- Bessel
 - function J_p , 484
 - modified function K_q , 485
- best approximation, 47, 64, 65, 396, 398
- big- \mathcal{O} notation, 482
- bilinear form, 16
- Bochner
 - integral, 7
 - theorem, 219
- Borel σ -algebra, 5
- Borel–Cantelli lemma, 158
- boundary-value problem
 - two dimensions, **58**, **372**
 - two-point, **40**
- Brownian
 - bridge, **193**
 - continuity of fBm, 211
 - existence Brownian motion, 211
 - fBm, **195**
 - motion, 181, 185, **185**
 - motion, two-sided, 187
 - sheet, 259
- Brownian motion, 319
- $C(D)$, 2
- $C(D, Y)$, $C^r(D, Y)$, 3
- $C^\infty(D, Y)$, $C_c^\infty(D, Y)$, 3
- Cameron–Martin space, 445
- Cauchy sequence, 2
- Cauchy–Schwarz inequality, 10, 149
- Céa’s lemma, 47, 64
- central limit theorem, 161
- CFL condition, 120, 129, 458
- Chapman–Kolmogorov equation, 348
- characteristic function, 146
- Chebyshev inequality, 158
- circulant embedding
 - approximate, **249**, **280**
 - extension, 244, 272
 - in two dimensions, **266**
 - minimal, 244
 - stationary processes, **241**
- classical solution, 41
 - two-point BVP, 43
- $CN(\mu, C)$, 227
- coloured noise, **196**
- compact
 - operator, 19
 - set, 19
- complete, 2
 - measure space, 5
- completion, 14
- conditional
 - expectation, 153
 - Gaussian distribution, 155
 - probability, 155
- continuity
 - realisations, **304**
 - sample path, **210**
- continuous function, 2, 3
- contraction mapping theorem, 2
- convergence
 - Euler for ODEs, 92
 - non-smooth error analysis, 124
 - order of, 91
 - pointwise, 2
 - random variables, **157**
 - SODE-strong, **337**
 - SODE-weak, **346**
 - SPDE-strong, **460**
 - spectral Galerkin, 48
 - testing numerically, 93, 117, 123, 343, 474
 - uniform, 2

- correlation, **142**
 - coefficient, 143
- covariance, 142, **189**
 - Bessel, 266, 285
 - cosine, 192
 - exponential, 208, 220, 232, 247, 248, 455
 - function, 185
 - Gaussian, 185, 208, 220, 232, 262
 - isotropic exponential, 263
 - matrix, 146
 - operator, 149
 - random field, 258
 - separable exponential, 261, 262
 - sinc, 234
 - Whittle–Matérn, 220, 264, 291, 298
- covariance matrix
 - BTTB, 276
 - stationary process, 275
- cylindrical Wiener process, 443, 457, 464
- δ_{ij} , *see* Kronecker delta function
- Daniel–Kolmogorov theorem, 190
- DFT, *see* Fourier
- Dini’s theorem, 21
- Dirac delta function, 13, 26
- distribution
 - binomial, 160
 - Cauchy, 141, 167
 - complex Gaussian, 227
 - Gaussian, 140
 - H -valued Gaussian, 150
 - joint, 142
 - log-normal, 142
 - multivariate Gaussian, 147
 - multivariate uniform, 145
 - uniform, 140
 - uniform on S^d , 167
- domain, 1
 - linear operator, 23
- dominated convergence theorem, 7
- Doob’s maximal inequality, 487
- Doob’s submartingale inequality, 305, 486
- Doob–Dynkin lemma, 154, 395
- Duffing–van der Pol, 317, 331
- eigenvalue, 19
- Euler method
 - ODEs, 92
- Euler method
 - explicit, 90
 - implicit, 90
 - semi-implicit, 96
 - stability, 95
- Euler–Maruyama, 330
 - θ -method, 336
 - stability, 335
- event, 138
- existence
 - elliptic PDE, 62
 - mild solution (deterministic), 104
 - mild solution (stochastic), 450
 - ODEs, 89
 - SODEs, 325
 - two-point BVP, 45
 - expectation, **138**, 139
 - conditional, **152**, 153
 - estimation, **162**
 - exponential
 - of linear operator, 101
 - exponential integrator, 136, 370, 470
 - \mathcal{F}_t -Brownian motion, 319
 - fBm, *see* Brownian motion
 - FFT, *see* Fourier
 - filtration, 319
 - finite difference method
 - reaction–diffusion equation, 107
 - SPDE, 455
 - finite element mesh
 - admissible, 66
 - shape-regular, 66
 - size, 66
 - finite element method (FEM), **40**, **50**, **66**
 - reaction–diffusion equation, **120**
 - basis function
 - global, 69
 - local, 71
 - diffusion matrix, 51
 - error analysis, 55, **77**, 377, 397
 - Galerkin matrix, 51, 71
 - global basis functions, 50, **69**
 - local basis functions, 52, **71**
 - mass matrix, 51, 71
 - non-smooth error estimate, 125
 - SPDE, **471**
 - finite-dimensional distribution, 184
 - finite-dimensional noise, 393
 - fluctuation–dissipation relation, 316
 - Fourier
 - analysis, **28**
 - discrete Fourier transform (DFT), **28**
 - fast Fourier transform (FFT), **29**
 - Hankel transform, 34
 - matrix, 28, 269
 - series, **29**
 - transform, **33**
 - two-dimensional DFT, 270
 - fractional powers, **25**
 - Frobenius norm, 17
 - Fubini theorem, 7
 - Galerkin
 - approximation, **40**, 46, 65, 110, 377, 378, 398
 - approximation (stochastic), **396**
 - finite element method (FEM), **66**
 - SPDE, 471
 - orthogonality, 47
 - projection, 47, 397
 - semilinear evolution equation, 112
 - stochastic PDE, 459
 - Galerkin approximation
 - semilinear PDEs, 110
 - Gamma function $\Gamma(x)$, 484
 - Gaussian, *see* distributions

- geometric Brownian motion, 318, 328
 Green's function, 23
 Gronwall inequality, 134
 H^r , H_0^1 , H_{per}^2 , 13
 $H^{1/2}(\partial D)$, 61
 $\mathcal{H}_{2,T}$, 324, 450
 Hankel transform, **34**
 heat equation
 Dirichlet boundary conditions, 99
 link to Brownian motion, 187
 on \mathbb{R} , 98
 spectral Galerkin, 113
 Hermitian vector, 242
 Heun's method, 134, 363, 481
 ODEs, 96
 Hilbert space, **9**
 valued random variables, **148**
 Hilbert–Schmidt
 operator, **17**
 spectral theorem, 20
 Hölder
 continuous, 4
 inequality, 9
iid, 160
 independence, **142**, 144
 indicator function, 4
 inner product, 9
 integral operator, 18
 isotropic, 34
 exponential covariance, 263, 287
 random field, 263
 Itô
 formula, 326, 350
 integral, 320, 445
 isometry, 324, 445
 Itô SODEs, **324**
 $J_p(r)$, *see* Bessel function
 Jensen's inequality, 37, 158, 486
 $K_q(r)$, *see* Bessel function
 Karhunen–Loève
 convergence, 202, 293, **390**, 392
 eigenvalue decay, **296**
 expansion, 221, **389**
 random fields, **293**, **389**
 stochastic process, **199**
 theorem, 202, 293
 uniform convergence, 203
 kernel, 18
 Kolmogorov equation, 189
 Kronecker delta function, 11
 Kronecker product, 269, 405
 \mathcal{L}_2^T , 323
 L^2 , 10
 $L^2(\Omega, \mathcal{F}, H)$, 153
 $L^2(\partial D)$, 60
 L_0^2 , 445
 L^p , 8
 $L_p^2(\Gamma, L^2(D))$, 394
 Lambert W function, 488
 Langevin equation, 317
 Langmuir equation, 103
 Laplacian with Dirichlet conditions, 22, 26, 49
 law of large numbers, 161
 law of random variable, 138
 Lax–Milgram lemma, 16
 least squares, **152**
 Lebesgue
 integral, **4**, **6**
 measure, **5**
 spaces, **8**
 left-continuous, 320
 Lévy areas, 333
 linear
 functional, 16
 operator, **15**
 linear interpolation, 209
 Lipschitz continuous, 4
 Markov process, 348
 martingale, 157
 matrix
 BCCB, **267**
 BTTB, **267**
 circulant, 242, 268
 non-negative definite, 20
 positive definite, 20
 Toeplitz, 242, 267
 mean, *see* expectation
 mean-square
 continuity, 207
 derivative, 207, 258
 differentiable, 224
 measurable function, 6
 measure
 complete, 5
 definition, 5
 probability, 138
 product, 7
 σ -finite, 5
 Mercer's theorem, 21
 mesh, 66
 shape-regular, 67
 uniform, 74
 method of lines, **107**
 mild solution
 deterministic, 104
 stochastic, 449
 Milstein method, 332
 SPDE, 464
 Minkowski inequality, 9
 modification, 210
 moments, 139
 estimation, **162**
 Monte Carlo, **170**
 convergence, **172**
 FEM error analysis, 385
 finite element method (FEM), **380**
 variance reduction, **175**
 multilevel Monte Carlo
 SODEs, 353

Nagumo equation
 deterministic, 109, 122
 stochastic, 433, 455
 natural filtration, 320
 Nemytskii, 105, 450
 non-negative definite, 190
 function, 218
 operator, **20**
 non-smooth error estimates, **124**
 norm, 1
 energy, 44, 397
 equivalence, 15
 Frobenius, 17
 Hilbert–Schmidt, 17
 matrix, 19
 Sobolev, 14
 vector, 1
 normal distribution, *see* Gaussian
 normed vector space, 1
 $\mathcal{O}(h)$, 482
 operator
 bounded, 15
 non-negative definite, **20**
 self-adjoint, 25
 Ornstein–Uhlenbeck, 315
 orthogonal, **10**
 projection, 10, 416, 418
 orthonormal
 basis, 11, 417
 \mathbb{P}_X , 138
 Parseval identity, 29, 33
 PDEs
 elliptic BVP, 58, **372**, 374
 semilinear, 102
 pdf, *see* probability density function
 periodic boundary conditions, 24
 Plancherel identity, 29
 Poincaré’s inequality, 14
 Poisson equation, 22
 polynomials
 complete, 399
 global, 399
 Hermite, 401
 Legendre, 401
 orthogonal, 400, 402
 tensor product, 399
 predictable, 323
 probability, **137**
 density function, 140
 distribution, 138
 measure, 138
 space, 137
 Q -Wiener process, 435, 436
 quasi-optimal approximation, 47, 65
 random fields, **257**, 258, **372**
 continuity, **304**
 Gaussian, 259
 isotropic, 263, **263**
 second-order, **258**
 stationary, 261, **261**

random number generators, **164**
 random variables, **138**
 complex Gaussian, 227
 complex-valued, 225
 convergence, **157**
iid, 160
 independent, 144
 $L^2(D)$ -valued, 260
 law, 138
 multivariate Gaussian, 163
 \mathbb{R}^d -valued, 145
 real-valued, **140**
 sampling, 164
 sums, **160**
 uncorrelated, 143, 146, 149
 random walk, 181
 realisation, 138, 258, 431
 reference element, 72
 Richardson extrapolation, 135
 Riesz representation theorem, 16
 S^d , 484
 sample path, 182, 190, 431
 continuity, 304
 sampling
 change of variables, **167**
 Q -Wiener process, 439
 random variables, **164**
 rejection, **168**
 via quadrature, 234
 second-order
 random field, 258
 stochastic process, 185
 self-adjoint, 25
 self-similarity, 195
 semigroup, **97**, 102
 C_0 , 99
 exponential, 101
 heat, 100
 infinitesimal generator, **97**
 theory, **100**
 semilinear evolution equation, 102, **104**
 finite difference method, 107
 semi-implicit Euler and spectral Galerkin, 113
 spectral Galerkin, 112
 stochastic, 447
 semi-norm, 1
 separable, 11
 σ -algebra, 4, 137
 sub $\sigma(X)$, 143
 Sobolev
 embedding theorem, 15
 spaces, **12**, 13
 spaces
 $C(D, Y)$, $C^r(D, Y)$, 3
 $C^\infty(D, Y)$, $C_c^\infty(D, Y)$, 3
 $H_0^1 H_{\text{per}}^2$, 14
 H^r , H_0^1 , H_{per}^2 , 13
 $\mathcal{H}_{2,T}$, 324, 450
 \mathcal{L}_2^T , 323
 L^2 , 10

- $L^2(\Omega, \mathcal{F}, H)$, 153
- L_0^2 , 445
- L^p , 8
- $L^p(\Omega, H)$, 149
- $L_p^2(\Gamma, L^2(D))$, 394
- $W^{r,p}$, 13
- spectral decomposition, **199**
- spectral distribution, density, 219
- spectral Galerkin, 48
 - non-smooth error estimate, 124
 - semi-implicit Euler, 113
 - semilinear evolution equation, **112**
 - SPDE, 466
- stability
 - ODEs, 95
 - SODEs, 335
- stationary
 - increments, 195
- stationary process, 217
 - complex Gaussian, **233**
 - complex-valued, **225**
 - Gaussian, **217**
 - mean-square differentiability, **224**
 - real-valued, 217
 - sampling by circulant embedding, 241
 - sampling by quadrature, **234**
- stochastic process
 - second-order, 185
- stochastic collocation, 421
- stochastic Galerkin
 - basis functions, 402
 - error analysis, **415**
 - finite element method (FEM), **396**
 - linear system, 404
- stochastic integral
 - Itô, **318, 445**
 - complex valued, 231
 - Fourier, 232
 - real valued, **228**
 - Stratonovich, **360**
- stochastic PDE
 - Allen–Cahn equation, 432, 468
 - heat equation, 447
 - Nagumo equation, 433
 - parabolic Anderson model, 450
 - vorticity, 434
- stochastic process, **181, 182**
 - complex-valued, 227
 - complex-valued Gaussian, 227
 - examples, **193**
 - Gaussian, **185, 189**
 - independent, 190
 - mean-square continuity, 207
 - regularity, **206**
 - sample path regularity, **210**
 - stationary, *see* stationary process
- Strang's lemma, 82
- strong solution, 41, 447
- submartingale, 304
- support, $\text{supp } u$, 3
- symmetric
 - operator, 19
- Taylor's theorem, 482
- tensor product space, 398
- trace class, 21
- trace operator (boundary), 60
- trapezium rule, 483
- turning bands, **283**
 - covariance, 286
 - operator, 284
- unbounded operator, **22**
- uncorrelated, *see* random variables
- uniform, *see* distributions
- variance, 139
 - estimation, **162**
- variance reduction, *see* Monte Carlo
- variational form
 - elliptic PDEs, **58, 64**
 - SPDE, 447
 - two-point boundary-value problem, **42**
- variational form (elliptic PDEs)
 - on $D \times \Gamma$, **393**
 - on $D \times \Omega$, **386**
 - on D , **374**
- version, 210
- wave equation, 135
- weak approximation SODEs, **346**
- weak derivative, 13
- weak solution
 - elliptic PDE, 61, 377, 378, **386, 387, 395, 396, 398**
 - semi-discrete, 396
 - semilinear PDEs, 112
 - SPDE, 447
 - two-point BVP, 43
- white noise, **196**
 - space–time, 443
- Wiener–Khintchine theorem, 219, 261
- Wong–Zakai theorem, 361