Introduction

So this book is going to be about the style of mathematics. Does it mean I am going to ignore the substance of mathematics? To some extent, I do, but then again not: the two dimensions are distinct, yet they are not orthogonal, so that stylistic preferences inform the contents themselves, and vice versa. For an example, I shall now take a central work of Hellenistic mathematics – Archimedes' *Spiral Lines* – and read it twice, first – very quickly – for its contents, and then, at a more leisurely pace, for its presentation of those contents. Besides serving to delineate the two dimensions of style and content, this may also serve as an introduction to our topic: for *Spiral Lines* is a fine example of what makes Hellenistic science so impressive, in both dimensions. For the mathematical contents, I quote the summary in Knorr 1986: 161 (fig. 1):

> The determination of the areas of figures bounded by spirals further illustrates Archimedes' methods of quadrature. The Archimedean plane spiral is traced out by a point moving uniformly along a line as that line rotates uniformly about one of its endpoints. The latter portion of the treatise *On Spiral Lines* is devoted to the proof that the area under the segment of the spiral equals one-third the corresponding circular sector . . . The proofs are managed in full formal detail in accordance with the indirect method of limits. The spirals are bounded above and below by summations of narrow sectors converging to the same limit of one-third the entire enclosing sector, for the sectors follow the progression of square integers. This method remains standard to this very day for the evaluation of definite integrals as the limits of summations.

Since I intend this book to be readable to non-mathematicians, I shall not try to explain here the geometrical structure underlying Knorr's exposition. Suffice for us to note the great elegance of the result obtained – precise numerical statements concerning the values of curvilinear, complex areas. Note the smooth, linear exposition that emerges with Knorr's summary, as if the spiral lines formed a strict mathematical progression leading to a quadrature, based on methods that in turn (in the same linear progression,

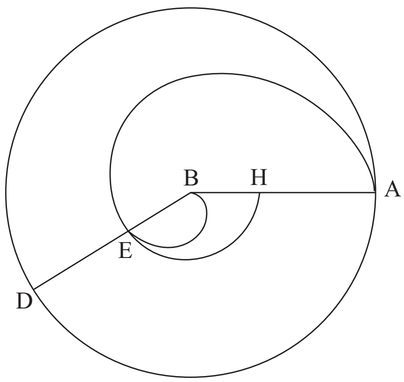

Figure 1

now projected into historical time) serve to inform modern integration. One bounds a problem – the spiral contained between external and internal progressions of sectors – and one then uses the boundaries to solve the problem – the progression is summed up according to a calculation of the summation of a progression of square integers. Such is the smooth, transparent intellectual structure suggested by Knorr's summary.

Let us see, now, how this treatise actually unfolds – so as to appreciate the achievement of Hellenistic mathematics in yet another, complementary way.

We first notice that the treatise is a letter, addressed to one Dositheus – known to us mainly as Archimedes' addressee in several of his works.[1] The social realities underlying the decision made by several ancient authors, to clothe their treatises as letters, are difficult for us to unravel. A lot must have to do with the poetic tradition, from Hesiod onwards, of dedicating the didactic epic to an addressee, as well as the prose genre of the letter-epistle as seen, e.g., in the extant letters of Epicurus.[2] The nature of the ancient mathematical community – a small, scattered group of genteel amateurs – may also be relevant.[3] In this book we shall return time and again to the literary antecedents of Hellenistic mathematics, as well as to its character as refined correspondence conducted inside a small, sophisticated group – but this of course right now is nothing more than a suggestion.

[1] See Netz 1998b for some more references and for the curious fact, established on onomastic grounds, that Dositheus was probably Jewish.
[2] I return to discuss this in more detail on pp. 104–5 below.
[3] See Netz 1999, ch. 7 for the discussion concerning the demography of ancient mathematics.

Let us look at the introduction in detail. Archimedes mentions to Dositheus a list of problems he has set out for his correspondents to solve or prove. Indeed, he mentions now explicitly – apparently for the first time – that two of the problems were, in fact, snares: they asked the correspondents to prove a *false* statement. All of this is of course highly suggestive to our picture of the Hellenistic mathematical exchange. But even before that, we should note the texture of writing: for notice the roundabout way Archimedes approaches his topic. First comes the general reminder about the original setting of problems. Then a series of such problems is mentioned, having to do with the sphere. Archimedes points out that those problems are now solved in his treatise (which we know as *Sphere and Cylinder* II), and reveals the falsity of two of the problems. Following that, Archimedes proceeds to remind Dositheus of a second series of problems, this time having to with conoids. We expect him to tell us that some of those problems were false as well, but instead he sustains the suspense, writing merely that the solutions to those problems were not yet sent. We now expect him to offer those solutions, yet the introduction proceeds differently:[4]

> After those [problems with conoids], the following problems were put forward concerning the spiral – and they are as it were a special kind of problems, having nothing in common with those mentioned above – the proofs concerning which I provide you now in the book.

So not a study of conoids, after all. We now learn all of a sudden – four Teubner pages into the introduction – that this is going to be a study of spirals. And we are explicitly told that these are "special," "having nothing in common" – that is, Archimedes explicitly flaunts the exotic nature of the problems at hand. We begin to note some aspects of the style: suspense and surprise; sharp transitions; expectations raised and quashed; a favoring of the exotic. No more than a hint of that, yet, but let us consider the unfolding of the treatise.

Now that the introduction proper begins, Archimedes moves on to provide us with an explicit definition of the spiral (presented rigorously but discursively as part of the prose of the introduction), and then asserts the main goals of the treatise: to show (i) that the area intercepted by the spiral is one third the enclosing circle; (ii) that a certain line arising from the spiral is equal to the circumference of the enclosing circle; (iii) that the area resulting from allowing the spiral to rotate not once but several times about the starting-point is a certain fraction of the enclosing circle,

[4] Heiberg 1913: 13–17.

defined in complex numerical terms; and finally (iv) that the areas bounded between spirals and circles have a certain ratio defined in a complex way. Following that Archimedes recalls a lemma he shall use in the treatise (used by him elsewhere as well, and known today, probably misleadingly, as "Archimedes' Axiom"). At this point the next sentence starts with εἴ κα κατά τινος γράμμας, "if on some line," i.e. without any particle, so that the reader's experience is of having plunged into a new sequence of prose and, indeed, the proofs proper abruptly begin here.

Before we plunge ourselves into those proofs, I have two interrelated comments on the introduction. The first is that the sequence of goals seems to suggest an order for the treatise, going from goal (i), through (ii) and (iii), to (iv). The actual order is (ii) – (i) – (iii) – (iv). The difference is subtle, and yet here is another example of an expectation raised so as to be quashed. The second is that the goals mentioned by Archimedes are put forward in the discursive prose of Greek mathematics of which we shall see many examples in the book – no diagram provided at this point, no unpacking of the meaning of the concepts. The result is a thick, opaque texture of writing, for example, the third goal:

> And if the rotated line and the point carried on it are rotated for several rotations and brought back again to that from which they have started out, I say that of the area taken by the spiral in the second rotation: the <area> taken in the third <rotation> shall be twice; the taken in the fourth – three times; the taken in the fifth – four times; and always: the areas taken in the later rotations shall be, by the numbers in sequence, multiples of the <area> taken in the second rotation, while the area taken in the first rotation is a sixth part of the area taken in the second rotation.

This is not the most opaque stated goal – the most opaque one is (iv). In fact I think Archimedes' sequence from (i) to (iv) is ordered in a sequence of mounting opaqueness, gradually creating a texture of prose that is heavy with difficult, exotic descriptions, occasionally rich in numerical terms. One certainly does not gain the impression that Archimedes' plan was to make the text speak out in clear, pedagogic terms.

This is also clear from the sequence of the proofs themselves. For no effort is made to explain their evolving structure. We were told to expect a treatise on measuring several properties of spirals, but we are first provided with theorems of a different kind. The first two propositions appear like physical theorems: for instance, proposition 2 shows that if two points are moved in uniform motions (each, a separate motion) on two separate straight lines, two separate times [so that altogether four lines are traced by

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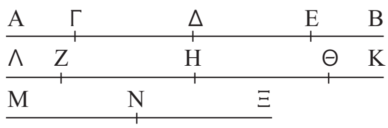

Figure 2

the two points each moving twice] the resulting lines are proportional. (See fig. 2.) Modern readers cannot but be reminded of Aristotle's *Physics*,[5] but for Archimedes' contemporary reader probably what came most to mind – as the scientific field where motions are discussed – was astronomy.[6] All the more surprising, then, that the motions discussed are *along a straight line* – i.e. related, apparently, neither to stars nor to spirals. It should be stressed that Archimedes simply presents us with the theorems, without a word of explanation of how they function in the treatise. So the very beginning does two things: it surprises and intrigues us by pointing in a direction we could not expect (theorems on linear motion!), and it underlines the fact that this treatise is about to involve a certain breaking down of the border between the purely theoretical and the physical. Instead of papering over the physical aspect of the treatise, Archimedes flags it prominently at the very beginning of the treatise. (I shall return to discuss this physical aspect later on.)

Do we move from theorems on linear motion to theorems on circular motion? This would be the logical thing to expect, but no: the treatise moves on to a couple of observations (not even fully proved) lying at the opposite end of the scientific spectrum, so to speak: from the physical theorems of 1–2 we move to observations 3–4 stating that it is possible in general to find lines greater and smaller than other given circles – the stuff of abstract geometrical manipulation. No connection is made to the previous two theorems, no connection is made to the spiral.

[5] See e.g. the treatment of the proportions of motion in such passages as *Physics* VII.5.

[6] By the time Archimedes comes to write *Spiral Lines*, Aristotle's Lyceum was certainly of relatively little influence. The texts of course were available (see Barnes 1997), but, for whatever reason, they had few readers (Sedley 1997 suggests that the very linguistic barrier – Attic texts in a *koine*-speaking world – could have deterred readers). On the other hand, it does appear that Archimedes admired Eudoxus above all other past mathematicians, and would probably expect his audience to share his admiration. (Introduction to *SC* 1, Heiberg 4.5, 11; introduction to *Method*, Heiberg 430.2, in both places implicitly praising himself for rising to Eudoxus' standard. No other past mathematician is mentioned by Archimedes in such terms.) Eudoxus was, among other things, the author of *On Speeds* (the evidence is in Simplicius, on Arist. *De Cael.* 488.3 ff.) – an astronomical study based on the proportions of motion. I believe this would be the natural context read by Archimedes' audience into the first propositions of *Spiral Lines*.

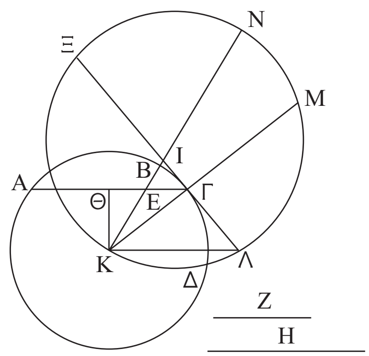

Figure 3

And immediately we switch again: if 1–2 were physical, while 3–4 were rudimentary geometrical observations, now we have a much richer sequence of pure geometry. So pure, that the relation to the spiral becomes even more blurred. Propositions 5–9 solve interesting, difficult problems in the geometry of circles, involving complex, abstruse proportions: for instance (proposition 8):

Given a circle and, in the circle, a line smaller than the diameter, and another, touching the circle at the end of the <line>[7] in the circle: it is possible to produce a certain line from the center of the circle to the <given> line, so that the <line> taken of it between the circumference of the circle and the given line in the circle has to the <line> taken of the tangent the given ratio – provided the given ratio is smaller than that which the half <line> of <line> given in the circle has to the perpendicular drawn on it from the center of the circle.

(In terms of fig. 3, the claim is that given a line in the circle AΓ and the tangent there ΞΛ, as well as the ratio Z:H, it is possible to find a line KN so that BE:BI::Z:H.) A mind-boggling, beautiful claim – of little obvious relevance to anything that went before in the treatise, or to the spirals themselves.

But this is as nothing compared to what comes next. For now comes a set of two propositions that do not merely fail to connect in any obvious way to the spirals – they do not connect obviously to anything at all. These are very difficult to define. Archimedes' readers would associate them with proportion theory, perhaps, or with arithmetic, but mostly they would consider those proofs to be *sui generis*. They would definitely consider

[7] Here and in what follows, text inside pointed brackets is my supplying of words elided in the original, highly economic Greek.

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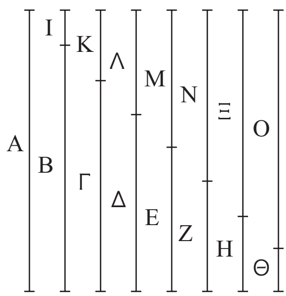

Figure 4

them enormously opaque. I quote the simpler enunciation among them, that of proposition 10:

> If lines, however many, be set consecutively, exceeding each other by an equal <difference>, and the excess is equal to the smallest <line>, and other lines be set: equal in multitude to those <lines mentioned above>, while each is <equal> in magnitude to the greatest <line among those mentioned above>, the squares on the <lines> equal to the greatest <i.e. the sum of all such squares>, adding in both: the square on the greatest, and the <rectangle> contained by: the smallest line, and by the <line> equal to all the <lines> exceeding each other by an equal <difference> – shall be three times all the squares on the <lines> exceeding each other by an equal <difference>.

In our terms, in an arithmetical progression $a_1, a_2, a_3, \ldots, a_n$ where the difference between the terms is always equal to the smallest a_1, the following equation holds:

$$(n+1)a_n{}^2 + (a_1 * (a_1 + a_2 + a_3 + \cdots + a_n)) \\ = 3(a_1{}^2 + a_2{}^2 + a_3{}^2 + \cdots + a_n{}^2)$$

This now makes sense, to some of us – but this is only because it is put forward in familiar terms, and such that serve to make the parsing of equations a lot easier.[8] The original was neither familiar to its readers nor spelled out in a friendly format. This was a take-it-or-leave-it statement of a difficult, obscure claim. And the proof does not get any easier. The addition of a diagram (fig. 4) certainly helps to parse the claim, but the operations are difficult, involving a morass of calculations whose thread is difficult to follow (I quote at random):

[8] It is also helpful to try and check the validity of the equation, so try this: with 1, 2, 3, 4 you have (5*16)+(1*10) = 3(1+4+9+16), which is in fact correct!

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and since two <rectangles> contained by B, I are equal to two <rectangles> contained by B, Θ . . . [a long list of similar equalities] and two <rectangles contained> by Δ, Λ are equal to the <rectangle contained> by Θ and the <line> six times Δ – since Λ is three times Δ . . . [a statement of a set of similar equalities, stated in a complex abstract way] – so all <rectangles taken twice>, adding in the <rectangle> contained by Θ and by the <line> equal to A, B, Γ, Δ, E, Z, H, Θ, shall be equal to the <rectangle> contained by Θ and by the <line> equal to all: A and three times B and five times Γ, and ever again, the following line by the odd multiple at the sequence of odd numbers.

We see that no effort is made to compensate for the obscurity of the enunciation by a proof clearly set out. The difficulty of parsing the statements is carried throughout the argument, serving to signal that this pair of propositions, 10–11, is a special kind of text, marked by its exotic nature.

We also notice that a new kind of genre-boundary is broken. If the first two propositions were surprising in their *physical* nature (having to do with a study of motions along lines), these two propositions 10–11 depend on calculation and in general suggest an arithmetical, rather than a geometrical context. One can say in general that Greek geometry is defined by its opposition to two outside genres. It is abstract rather than concrete, marking it off from the physical sciences, while, inside the theoretical sciences, it is marked by its opposition to arithmetic.[9] Archimedes, in this geometrical treatise, breaks through the genre-boundaries with *both* physics and arithmetic.

And yet this is geometry, indeed the geometry of the spiral. We were almost made to forget this, in the surprising sequence going from physics, through abstract, general geometrical observations, via the geometry of

[9] In Aristotle's architecture of the sciences we often see the exact sciences as falling into geometrical and arithmetical, in the first place, and then the applied sciences related to them (e.g. music to arithmetic, optics to geometry: *An. Post.* 75b15–17, 78b37–9). Plato's system of *mathēmata* famously (*Resp*. 525a–531d) includes arithmetic, geometry, astronomy and music, with stereometry uneasily accommodated: since astronomy is explicitly related to stereometry, Plato would presumably have meant his audience to keep in mind the relationship of music to arithmetic, though he merely points out in the conclusion of this passage that the relationship between the sciences is to be worked out (530c9–d4), and he does echo the Pythagorean notion of "sisterhood" of astronomy and music (they are, more precisely, *cousins*) – perhaps derived from Archytas' fr. 1 l.7 (Huffman 2005: 11.1.). It is not clear that anyone in Classical Antiquity other than Aristotle would have explicitly objected to the mixing of scientific disciplines (Late Antiquity is of course already much more self-conscious of such boundaries; it is curious to note that, when Eutocius makes an apology for what he perceives to be a potentially worrisome contamination of geometry by arithmetic – *In Apollonii Conica*, Heiberg 1893: 11.220 ll. 17–25 – he relies explicitly on the notion of the sisterhood of the disciplines!); the same is true, in fact, for literary genres. It may well be that the explicit notion of a scientific discipline – as well as a literary genre – in the age of Aristotle could, paradoxically, facilitate the hybridization of genres characteristic of the third century. But on all of this, more below.

circles and tangents, and finally leading on to a *sui generis* study of arithmo-geometry – none of these being relevant to any of the others. Yet now – almost halfway through the treatise – we are given another jolting surprise. All of a sudden, the text switches to provide us with its proper mathematical introduction! And we now have the explicit definitions of the spiral itself and of several of the geometrical objects associated with it. One is indeed reminded of how we have learned only well into the introduction that this treatise is going to be about spirals, but the surprise here is much more marked, as the very convention of a geometrical introduction is subverted, rather like Pushkin remembering to address his muse only towards the *end* of the first canto of his *Eugene Onegin*. Of course this belated introduction now serves to mark the text and divide it: what comes before is strictly speaking introductory, and the geometry of the spiral itself now unfolds in the following propositions.

And what a mighty piece of work this geometry now is! Having put behind us the introductory material with its ponderous pace, the treatise now proceeds much more rapidly, quickly ascending to the results proved by Archimedes in his introduction. It takes surprisingly little effort, now, to get to goal (ii), where a certain straight line defined by the spiral is found to be equal to a circumference of the circle. Quite a result, too: for after all this is a kind of squaring of the circle. Archimedes obtains this in proposition 18, in a proof that directly depends upon the subtle problems 5–9 having to do with tangents to circles, and indirectly depends upon the first two, physical theorems – which therefore now find, retrospectively, their position in the treatise. And, contrary to the expectation established in the introduction, he does not see proposition 18 as a conclusion for its line of inquiry. Proposition 18 determines an equality between a circumference of a circle and a line produced by a *single* rotation of the spiral. Proposition 19 then shows that the same line produced by the second rotation of the spiral is twice a given circumference, and moves on to generalize to the much more striking (and arithmetized) result, that the rotation of a given number produces a line which is as many times as the given number the circumference of the circle, while the following proposition 20 moves on to show a similar result for a different type of straight line. The single goal (ii) sprouts into an array of results, heavily arithmetical in character.

At this point a new attack on the spiral develops – the most central one, giving rise to the measurement of the area of the spiral. But the reader of Archimedes would be hard pressed to know this.

Archimedes has just proved goal (ii) set, far back, in the introduction, and then went on to add further consequences, not even hinted at in that introduction. At this point, therefore, the reader is thoroughly disoriented: the next proofs can be about some further consequences of goal (ii), or about goal (iii), (i), or anything else. There is no way for us to know that now begins, in effect, the kernel of the book.

The introduction of a further conceptual tool also serves to mystify. In proposition 21 Archimedes starts by introducing the notion of bounding the spiral area between sectors of circles – no suggestion made of how and where this fits into the program of the treatise. And instead of moving on to utilize this bounding, Archimedes moves on in proposition 22 to generalize it to the case where the spiral rotates *twice*, and then generalize further in the same proposition to any number of rotations; in proposition 23 he generalizes it to the case of a partial rotation. Archimedes could easily have followed the argument for a single rotation, and only then generalize it, in this way making the conceptual structure of the argument somewhat less obscure. He has made the deliberate choice not to do so. In fact the reader by now may well think that Archimedes has plunged into a discussion of the relationship between spiral areas and sectors of circles – so extended is the discussion of the bounding of spirals between sectors, and so little outside motivation is given to it.[10]

Then we reach proposition 24 and now – only now! – the treatise as a whole makes sense, in a flash as it were.

The enunciation, all of a sudden, asserts that the spiral area is one-third the enclosing circle (this is said in economic, crystal-clear terms – the first simple, non-mystifying enunciation we have had for a long while). No mention is made in the enunciation of the sectors of circles. The proof starts by asserting that if the area is not one-third, it is either greater or smaller. Assume it is smaller, says Archimedes – and then he recalls the enclosing sectors. And then he notes almost in passing – in fact, he uses the expression "it is obvious that" – that the figure made of enclosing sectors instantiates the result of proposition 10, *a result which only now is provided with its meaning in the treatise*. It then follows immediately that the assumption that the spiral area is smaller than one-third the circle ends up with the enclosing sectors as both greater and smaller than one-third the circle. The analogous result is then quickly shown for the case that the

[10] A mathematically sophisticated reader would no doubt identify the potential of such limiting sectors for an application of what, in modern literature is called the "method of exhaustion." But Archimedes does not assert this at any point, and what is even more important, no hint is provided as to how such sectors can be measured and so serve in the application of this method. This will be made clear in retrospect only, as Archimedes would soon reveal the relation of this sequence of sectors to proposition 10.