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### Sets and sequences

We denote by  $\mathcal{P}(A)$  the set of subsets of a set  $A$  (its powerset), and by  $\mathcal{P}_f(A)$  its set of finite subsets. The difference of two sets  $A$  and  $B$  is the set  $A - B := \{a \in A \mid a \notin B\}$ . The empty set is  $\emptyset$ .

We denote by  $Seq(A)$  the set of finite sequences of elements of a set  $A$ , and by  $s[i]$  the  $i$ -th element of  $s \in Seq(A)$ . The empty sequence is  $()$ . If  $A$  is an alphabet, we denote  $Seq(A)$  also by  $A^*$ . Its elements are then called words, which are sequences of letters (or symbols). The empty word is also denoted by  $\varepsilon$ .

We denote by  $|A|$  the cardinality of a set  $A$ , and also by  $|s|$  the length of a sequence  $s$  (in particular, of a word). The cardinality of a set  $A$  is also denoted by  $Card(A)$  in certain cases, for better readability of formulas. We denote by  $|s|_a$  the number of occurrences of  $a \in A$  in a sequence  $s \in Seq(A)$ .

### Integers

We denote by  $\mathcal{Z}$  the set of integers, by  $\mathcal{N}$  the set of nonnegative ones and by  $\mathcal{N}_+$  the set of positive ones. For  $n, m \in \mathcal{Z}$ , we let  $[n, m] := \{i \in \mathcal{Z} \mid n \leq i \leq m\}$  and  $[m] := [1, m]$ . We have  $[m] = \emptyset$  if  $m \leq 0$  and  $[n, m] = \emptyset$  if  $m < n$ .

If  $p, q \in \mathcal{N}$ ,  $q \geq 2$ , we let  $\text{mod}_q(p)$  be the unique integer  $r$  in  $[0, q - 1]$  such that  $p \equiv r \pmod{q}$ . If  $n \in \mathcal{N}$ , then  $\text{exp}(n)$  denotes  $2^n$ . All logarithms are in base 2. The function  $\text{exp} : \mathcal{N}^2 \rightarrow \mathcal{N}$  is defined by  $\text{exp}(0, n) = n$  and  $\text{exp}(d + 1, n) = 2^{\text{exp}(d, n)}$ ; thus,  $\text{exp}(1, n) = \text{exp}(n)$ .

### Binary relations and functions

If  $R \subseteq A \times A$  is a binary relation on a set  $A$ , then  $R^*$  denotes its reflexive and transitive closure,  $R^+$  its transitive closure and  $R^{-1}$  its inverse  $\{(x, y) \mid (y, x) \in R\}$ . The identity relation  $\{(x, x) \mid x \in A\}$  is denoted by  $Id_A$ , or just by  $Id$  when  $A$  is clear from the context.

If  $R \subseteq A \times B$  and  $S \subseteq B \times C$  are two binary relations, then  $R \cdot S$  denotes their composition, i.e., the relation  $\{(x,z) \in A \times C \mid (x,y) \in R \text{ and } (y,z) \in S \text{ for some } y \in B\}$ . If  $R$  and  $S$  are functional, i.e., if they define partial functions  $f : A \rightarrow B$  and  $g : B \rightarrow C$  respectively, then  $R \cdot S$  defines the partial function  $g \circ f : A \rightarrow C$ ; in that case we denote  $R \cdot S$  also by  $S \circ R$ . We denote by  $\bigcirc_{i \in I} f_i$  the composition in any order of functions  $f_i$  that commute pairwise (i.e., such that  $f_i \circ f_j = f_j \circ f_i$  for all  $i, j \in I$ ).

The domain of a binary relation  $R \subseteq A \times B$  is denoted by  $Dom(R) \subseteq A$ , and its image is  $R(A) \subseteq B$ . Thus,  $Dom(R) = \{a \in A \mid (a,b) \in R \text{ for some } b \in B\} = R^{-1}(B)$ . For a subset  $C$  of  $A$ ,  $R(C) = \{b \in B \mid (a,b) \in R \text{ for some } a \in C\}$ .

The restriction of a mapping  $f : A \rightarrow B$  to a subset  $C$  of  $A$  is denoted by  $f \upharpoonright C$ . Two mappings  $f : A \rightarrow B$  and  $g : A' \rightarrow B'$  agree if  $f(a) = g(a)$  for every  $a \in A \cap A'$ , i.e., if  $f \upharpoonright (A \cap A') = g \upharpoonright (A \cap A')$ . We denote by  $f \cup g$  their common extension into a mapping  $: A \cup A' \rightarrow B \cup B'$ .

We denote by  $[A \rightarrow A]_f$  the set of mappings  $: A \rightarrow A$  that are the identity outside of a finite subset of  $A$  and by  $Perm_f(A)$  the subset of those that are permutations, i.e., bijections  $: A \rightarrow A$ .

## Other symbols

Notation that is self-explanatory (e.g.,  $Loops(G)$  for the set of loops of a graph  $G$ ), or that is used in a single section, is not listed. The order below is conceptual: concepts that are close mathematically are put together as closely as possible. General concepts are given before the more technical ones. Symbols are followed by short explanations and the page numbers where they are defined.

### *Terms and rooted trees (Sections 2.1 and 2.6.1, Definitions 2.13 and 2.14)*

$F, F_i, F_+$	functional signature	81, 177
$F^-$	functional signature	202
$\rho(f), \rho(F)$	arity	81, 177
$H \subseteq F$	subsignature	81, 177
$T(F), T(F, X)$	sets of terms	82, 84, 178
$T(F)/\equiv_{AC(H)}$	terms (AC signature)	412
$Slim(F)$	slim terms	143
$Pos(t), Occ(t, f)$	positions, occurrences	83
$ht(t)$	height of term $t$	83
$ListVar(t), \tilde{t}$	variables in $t$	85
$t[t_1/v_1, \dots, t_n/v_n], c[t]$	substitutions	84
$\theta_H$	second-order substitution	179
$Ctxt(F), Ctxt(F, X)$	contexts over $F$	84
$t \uparrow u$	context above $u$	84
$t/u$	subterm below $u$	84

Cambridge University Press

978-0-521-89833-1 - Graph Structure and Monadic Second-Order Logic: A Language-Theoretic Approach

Bruno Courcelle and Joost Engelfriet

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$N_T, \text{root}_T$	nodes, root of tree $T$	91
$\text{son}_T, \leq_T$	son, ancestor in $T$	92
$T/u$	subtree below $u$	91
$\text{Syn}(t), N_t, \leq_t$	syntactic tree of $t$	92, 94
$\text{Syn}(t)/u$	$\simeq$ syntactic tree of $t/u$	93
$\text{Syn}(t) \uparrow u$	$\simeq$ syntactic tree of $t \uparrow u$	93
$N_t/u, N_t \uparrow u$	nodes of $\text{Syn}(t)/u, \text{Syn}(t) \uparrow u$	94
$\text{yd}(t)$	yield of $t$	587

*Algebras (Sections 2.1 and 2.6.1)*

$\mathbb{M}$	$F$ -algebra	81, 176
$M$	its domain	81, 177
$f_{\mathbb{M}}$	operation defined by $f$	81, 176
$\mathbb{T}(F), \mathbb{T}_{AC}(F, H)$	algebras of terms	82, 178, 412
$\mathbb{W}(A), \mathbb{W}_{\text{left}}(A), \mathbb{W}_{\text{right}}(A)$	algebras of words	85
$F_A, U_A$	their signatures	85
$\text{val}_{\mathbb{M}}(t)$	value of term $t$	82, 178
$t_{\mathbb{M}}$	derived operation	82, 84, 178, 179
$\mathcal{S}$	set of sorts	176
$M_s$	domain of sort $s$	176
$M_{(s_1, \dots, s_n)}$	product of domains	176
$T(F)_s, T(F, X)_s$	terms of sort $s$	178
$\alpha(f), \sigma(f)$	input, output type of $f$	176
$\sigma(m)$	sort of $m$	177
$s_1 \times \dots \times s_n \rightarrow s$	type of an operation	176
$\mathbb{M} \subseteq \mathbb{N}$	subalgebra	81, 177
$\mathbb{M} \times \mathbb{N}$	Cartesian product	82, 177
$( A , \text{enc}_A, \xi_A), \tilde{f}$	encodings	85, 86
$\zeta_{\mathbb{M}}$	operations of $\mathbb{M}$	86
$\mathbb{M} \upharpoonright F, \mathbb{M} \upharpoonright \mathcal{S}$	restriction	177
$\mathbb{M}_H$	derived algebra	179
$\mathbb{M}/\sim$	quotient algebra	233
$\mathbb{N}^k$	free commutative monoid	205, 255

*Graphs (Section 2.2)*

$V_G, E_G$	vertices, edges of graph $G$	87
$\text{vert}_G$	incidence mapping	87
$\text{edg}_G$	adjacency relation	88
$u \rightarrow_G v, u -_G v$	directed, undirected edge	87
$\text{Isol}_G$	isolated vertices	107
$\text{Adj}^-(x)$	adjacent vertices	133, 657

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$\emptyset$	empty graph	89
$\ G\ $	size of $G$	89
$Deg(G)$	(maximal) degree of $G$	53
$(K, \Delta)$	vertex and edge labels	90
$G \simeq H$	isomorphism of graphs	89
$[G]_{iso}$	isomorphism class of $G$	89
$Inc(G)$	incidence graph	69, 126
$und(G), core(G)$	undirected graphs from $G$	88
$und(G)^k$	distance $k$ graph	551
$Line(G)$	line graph of $G$	71
$G \cup H, G \cap H$	union, intersection	99
$G \oplus H$	disjoint union	99
$G \otimes H$	complete join	18
$H \subseteq G, H \subseteq_i G$	subgraph, induced subgraph	91
$G[X], G - X, G - u$	subgraphs	91
$G[F], G - F, G - e$	subgraphs	91
$G/\approx$	quotient graph	94, 95
$G/F, \triangleleft, \triangleleft_c$	edge contraction, minor	95
$Forb(A), Obst(A)$	forbidden/excluded minors	98
$\omega(G)$	clique number	133
$K_n, K_{n,m}, C_n$	standard graphs	41, 125
$G_{n \times m}, G_{n \times m}^u, P_n$	grids, paths	125, 159
$wd$	width	108, 122, 136
$twd, pwd$	tree-width, path-width	122, 126
$TWD(\leq k, C), TWD(\leq k)$	tree-width $\leq k$	122
$PWD(\leq k, C), PWD(\leq k)$	path-width $\leq k$	126
$G[u \leftarrow H], G[e \leftarrow H]$	graph substitution	161, 278
$B(G), B_\Delta(G)$	encodings of graph $G$	437, 438
$\mathcal{G}^u$	simple undirected graphs	18
$\mathcal{G}$	simple directed graphs	46
$\mathcal{J}_2^d$	directed graphs with two sources	20

*Graph algebras: graphs with sources (Section 2.3)*

$\mathcal{JS}, \mathcal{JS}^u, \mathcal{JS}^d, \mathcal{JS}^t$	HR algebras of s-graphs	104, 105, 181
$\mathcal{JS}, \mathcal{JS}^u, \mathcal{JS}^d, \mathcal{JS}^C$	their domains	100, 104, 105, 181
$F^{HR}, F^{HRu}, F^{HRd}, F^{tHR}$	their operations	104, 105, 181
$F_C^{HR}$	subsignature for $C \subseteq \mathcal{A}$	105
$\mathcal{JS}[C], \mathcal{JS}^{gen}[C]$	subalgebras	105, 110, 181
$P_C^{HR}, F_C^{HRd}$	derived signatures	143, 479
$\mathcal{J}_2^d, \mathcal{J}_2^d$	a derived subalgebra	20, 46, 184
$\mathcal{JN}^t, \mathcal{JN}$	a derived subalgebra	288

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$\mathcal{A}$	source names/labels	100
$\tau(G)$	type of s-graph $G$	100
$\tau(t), \widehat{\tau}(t), \mu(t), wd(t)$	related notions	108
$G^\circ$	s-graph without sources	100
$src_G, slab_G, Src(G)$	source designations	100
$Int_G$	internal vertices of $G$	100
$\beta(G), Int(G)$	border, interior of $G$	298
$G/\approx$	quotient s-graph	102
$\parallel, \square_k$	parallel-composition	101, 102, 288
$\parallel_{C,D}$	typed parallel-composition	181
$fg_B, fg_b$	source forgetting	102, 103
$fg_{B,C}$	typed source forgetting	181
$miv_a$	making internal vertex	479
$ren_h, ren_{a \rightarrow b}, ren_{a \leftrightarrow b}$	source renaming	103
$ren_{h,C}$	typed source renaming	181
$\mathbf{a}, \mathbf{a}^\ell, \mathbf{ab}, \mathbf{ab}, \emptyset$	constant symbols	103, 104
$\mathbf{a}^\ell_\lambda, \mathbf{ab}_\lambda, \mathbf{ab}_\lambda, \mathbf{a}_\kappa$	symbols for labeled graphs	105
$\bullet$	series-composition	20, 183
$val(t)$	abstract value of term $t$	104, 110
$cval(t), gval(t)$	concrete value of $t$	112, 114
$Exp(t)$	expansion of $t$	111
$Typ, Unt$	typing, untyping	181, 182, 272

*Graph algebras: graphs with ports (Section 2.5)*

$\mathbb{G}\mathbb{P}, \mathbb{G}\mathbb{P}^u, \mathbb{G}\mathbb{P}^t$	VR algebras of p-graphs	146, 185
$\mathcal{G}\mathcal{P}, \mathcal{G}\mathcal{P}^u, \mathcal{G}\mathcal{P}_C$	their domains	145, 184
$F^{VR}, F^{VRu}, F^{tVR}$	their operations	146, 185
$F_C^{VR}$	subsignature for $C \subseteq \mathcal{A}$	146
$\mathbb{G}\mathbb{P}[C], \mathbb{G}\mathbb{P}^{\text{gen}}[C]$	subalgebras	146, 185
$F^{cVR}$	concrete version of $F^{VR}$	150
$F^{iVR}, F^{iVR}, F^{\kappa VR}$	variants of $F^{VR}$	170–172
$\mathbb{G}^u, \mathcal{G}^u$	a derived subalgebra	18, 46
$\mathcal{A}$	port labels	144
$\pi(G)$	type of p-graph $G$	144
$G^\circ$	p-graph without ports	144
$port_G$	port mapping	144
$\oplus$	disjoint union	99, 145
$\oplus_{C,D}$	typed disjoint union	185
$\overrightarrow{add}_{a,b}, add_{a,b}$	edge addition	145
$\overrightarrow{add}_{a,b,C}, add_{a,b,C}$	typed edge addition	185

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$relab_h, relab_{a \rightarrow b}, relab_{D \rightarrow a}$	port relabeling	145, 161
$relab_{h,C}$	typed port relabeling	185
$\mathbf{a}, \mathbf{a}^\ell, \mathbf{a}(x), \mathbf{a}^\ell(x), \emptyset$	constant symbols	146, 150
$\overrightarrow{add}_{a,b,\lambda}, \mathbf{a}_M$	for labeled graphs	149
$ADD_R$	derived edge addition operation	155
$\otimes, \overrightarrow{\otimes}, \otimes_{R,h}$	complete join	18, 147, 161, 172
$val(t), cval(t)$	abstract, concrete value of $t$	146, 150, 151
$cval(t)/u$	$\simeq$ concrete value of $t/u$	151
$Occ_0(t), port_t$	notions for $cval(t)$	150, 151
$cwd, lcwd$	clique-width, linear $cwd$	148
$CWD(\leq k, C), CWD(\leq k)$	clique-width $\leq k$	295

*Equational and recognizable sets; automata (Chapters 3, 4 and 6)*

$\mathcal{P}(\mathbb{M})$	powerset algebra	189
$\cup_s$	union of sets of sort $s$	189
$\Omega_s$	empty set of sort $s$	189
$F_\cup$	signature of $\mathcal{P}(\mathbb{M})$	189
$Pol(F, X)$	polynomial terms	190
$Mon(t)$	monomials of $t$	190
$\mathcal{P}_h$	set extension of homomorphism $h$	191
$Unk(S)$	unknowns of system $S$	191
$Sort(S), F(S)$	sorts, operations in $S$	192
$S_{\mathcal{P}(\mathbb{M})}$	function associated with $S$	193
$\mu_{\bar{x}} \cdot S_{\mathcal{P}(\mathbb{M})}(\bar{x})$	least solution of $S$ in $\mathcal{P}(\mathbb{M})$	194
$\mu_{\bar{x}} \cdot S_{\mathcal{P}(\mathbb{M})}(\bar{x}) \upharpoonright y$	its component defined by $y$	194
<b>Equat</b> ( $\mathbb{M}$ )	equational sets of $\mathbb{M}$	194
<b>Rat</b> ( $\mathbb{M}$ )	rational sets of $\mathbb{M}$	221
$\Rightarrow_G, \Rightarrow_S$	one-step derivation relations	197, 207
$L(G, x), L(G, Y)$	context-free languages	197
$G[S], G(S), G(\mathcal{A})$	context-free grammars	197, 200, 223
$S \nearrow S'$	unfolding	208
$Trim(S), Trim(S, Z)$	trim systems obtained from $S$	212
$\langle F, Q_{\mathcal{A}}, \delta_{\mathcal{A}}, Acc_{\mathcal{A}} \rangle$	$F$ -automaton $\mathcal{A}$	222
$f[q_1, \dots, q_k] \rightarrow_{\mathcal{A}} q$	transition	222
$\delta_f$	transition function	582
$\sharp_{\mathcal{A}}, \ \mathcal{A}\ $	size measures	222
$L(\mathcal{A}, q), L(\mathcal{A})$	languages recognized by $\mathcal{A}$	223
$S(\mathcal{A})$	equation system of $\mathcal{A}$	223
$run_{\mathcal{A}, t}$	run of deterministic complete $\mathcal{A}$	224
$td\text{-run}_{\mathcal{A}, t}$	top-down run of $\mathcal{A}$	582
$Q_{\mathcal{A}, s}$	states of sort $s$	225

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$det(\mathcal{A})$	determinization of $\mathcal{A}$	225
$\mathbf{Rec}(\mathbb{M})$	recognizable sets of $\mathbb{M}$	228
$\mathcal{A}(\mathbb{B}, C)$	automaton from algebra	230
$\sim, (\sim_s)_{s \in S}$	congruence	233
$\gamma_{\sim}$	its index	233
$\mathbb{M}/\sim$	quotient algebra	233
$\approx^L$	syntactic congruence of $L$	234, 287, 305
$\gamma_L$	recognizability index of $L$	234, 287, 305
$\mathcal{M}(L)$	minimal automaton of $L$	237, 238
$(\widehat{p})_{p \in P}$	family of predicates	245
$sat(\mathbb{M}, p)$	satisfaction set	246
$L(S, x)$	s-graphs or p-graphs defined by $(S, x)$	262, 293
$L_{Term}(S, x)$	terms defined by $(S, x)$	262, 293
$A(S)$	source or port labels in $S$	262, 293
$\mathcal{A} \times \mathcal{B}$	product of automata	440
$\mathcal{A} \cup \mathcal{B}, \mathcal{A} \cap \mathcal{B}$	union, intersection of automata	440
$\overline{\mathcal{A}}$	complement automaton	440
$h(\mathcal{A}), h^{-1}(\mathcal{A})$	images of automaton $\mathcal{A}$	441

*Relational structures (Chapters 5, 7 and 9)*

$\mathcal{R}, \mathcal{R}_i, etc.$	relational signatures	
$\rho(R), \rho(\mathcal{R})$	arity	316
$\langle D_S, (R_S)_{R \in \mathcal{R}_+}, (c_S)_{c \in \mathcal{R}_0} \rangle$	$\mathcal{R}$ -structure $S$	316
$\emptyset, \emptyset_{\mathcal{R}}$	empty structures	316
$S[X], S/\approx$	substructure, quotient structure	316, 394
$S \simeq T$	isomorphism of structures	316, 510
$[S]_{iso}$	isomorphism class of $S$	316
$STR^c(\mathcal{R}), STR(\mathcal{R})$	concrete, abstract $\mathcal{R}$ -structures	316
$\mathcal{W}_A, suc, lab_a$	representation of words $w$	316
$\mathcal{R}_F, son_i, lab_f, rt,$	representation of terms $t$	317
$\mathcal{R}_s^{sof}, \mathcal{R}_{s,C}^{bri}, \mathcal{R}_{ss,C}$	signatures for (p-, s-) graphs $G$	320
$\mathcal{R}_{s,[K,\Delta]}, \mathcal{R}_m^u, etc.$		318, 319,
$edg, edg_{\lambda}, lab_a, in,$	relations for (p-, s-) graphs	345, 346
$in_1, in_2, lab_{Edge}$		318, 319,
$[w], [t], [G]_C, [G],$	structures representing $w, t, G$	345, 346
$[G]_C, [G]$		317–320,
$[t]/u, [t] \uparrow u$	representation of subterm,	345, 346
	context	321
$[L], [L], [L]_C, [L]_C$	structures for a set $L$	318–321,
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$\  \cdot \ $	size functions	322, 494, 649
$\mathcal{R}_*, Lab_{\mathcal{R}}, S_*, STR_*^c(\mathcal{R}_*)$	constants as unary relations	328, 511
$\oplus, \oplus_{\mathcal{R}, \mathcal{R}'}$	disjoint union of structures	359, 376
$+$	union of structures	651
$\  \mathcal{R}, \mathcal{R}'$	parallel composition of structures	385
$fuse_{a,b}, ffuse_{a,b}, qf, add, mdf$	unary operations on structures	376, 377, 394, 643
$STR, STR_{pres}, STR_{sep}, STR_{nc}$	algebras of relational structures	394
$F^{QF}, F_{pres}^{QF}, F_{sep}^{QF}, F_{nc}^{QF}$	their functional signatures	394
$F^{QF} \upharpoonright \mathcal{R}$	subsignature	638
$F_C^{relIVR}, F_{\mathcal{R}, \mathcal{B}}^{redQF}$	related functional signatures	643, 647
$Inc(S), \mathcal{R}^{Inc}, In_{\mathcal{R}}, T_S$	incidence graph of $S$	629, 630
$twd(S), pwd(S)$	tree-width, path-width of $S$	632
$TWD(\mathcal{R}, \leq k), PWD(\mathcal{R}, \leq k)$	$\mathcal{R}$ -structures of $twd, pwd \leq k$	632
$Adj(S)$	adjacency graph of $S$	633
$twd^{Inc}(S)$	tree-width of $Inc(S)$	633
$cval(t), Occ_0(t)$	concrete structure defined by $t$	395, 638

*Logic (Chapters 5 and 6)*

Standard logical notation is not reviewed.

$ \varphi , qh(\varphi)$	size, quantifier-height of formula $\varphi$	324
$mfv(\varphi)$	maximal free variable of $\varphi$	445
$bqh(\varphi)$	block quantifier-height of $\varphi$	447
$ba(\varphi)$	Boolean arity of $\varphi$	447
$\varphi[\dots]$	substitutions in $\varphi$	332, 333
$\varphi \upharpoonright X$	relativization	334
$TC[R; x, y], TC[(\lambda u, v \cdot \varphi); x, y]$	transitive closure	334, 335
$QF(\mathcal{R}, \mathcal{X})$	quantifier-free formulas over $\mathcal{R}, \mathcal{X}$	372
$FO(\mathcal{R}, \mathcal{X})$	first-order formulas	322, 323
$MS(\mathcal{R}, \mathcal{X})$	monadic second-order formulas	326
$SO(\mathcal{R}, \mathcal{X})$	second-order formulas	324
$FO_i, MS_i, SO_i, i = 1, 2$	formulas expressing graph properties	347
$x:v, x:e, X:v, X:e$	typed variables	349
$Sgl$	singleton set predicate	445
$Card_{p,q}$	cardinality set predicates	352, 353
$CMS(\mathcal{R}, \mathcal{X}), CMS_1, CMS_2$	counting MS logic	352, 353



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$\text{CMS}^{\text{Bool}}$	CMS logic with Boolean set terms	459
$\text{MOD}(\varphi)$	models of $\varphi$	323
$\text{Th}_{\mathcal{L}}(\mathcal{C})$	$\mathcal{L}$ -theory of $\mathcal{C}$	329
$\text{Sat}_{\mathcal{L}}(\mathcal{C})$	formulas of $\mathcal{L}$ satisfiable in $\mathcal{C}$	329
$\text{MC}(\mathcal{C}, \mathcal{L})$	model-checking problem	429
$\text{sat}(S, \varphi, \bar{X}, \bar{x})$	satisfying assignments	324
$\sharp\text{sat}(S, \varphi, \dots)$	counting function	432
$\text{Maxsat}(S, \varphi, \dots)$	optimizing function	432
$\text{sat}(S \oplus T, \varphi, \bar{X}, \bar{x}; \bar{y}), \text{perm}_{\pi}, \boxtimes$	notions for the Splitting Theorem	361, 365, 366
$\langle \delta, (\theta_R)_{R \in \mathcal{R}'_+}, (\kappa_{c,d}) \rangle$	QF operation definition scheme $\mathcal{D}$	373
$\widehat{\mathcal{D}}$	QF operation defined by $\mathcal{D}$	373
$\iota_{\mathcal{R}, \mathcal{R}'}, \varphi^{\mathcal{D}}$	natural inclusion backwards translation	375 380
$\text{Th}(S, \mathcal{R}, \cdot)$	bounded theory	388
$\text{Th}(t, \downarrow u, \cdot), \text{Th}(t, \uparrow u, \cdot)$	bounded theories for term $t$	390, 392
$\text{Th}_t^{\downarrow}(\cdot), \text{Th}_t^{\uparrow}(\cdot), \text{Th}_t(\cdot)$	special cases	583
$\text{Th}_{rt}^{\uparrow}$	root theory	583
$\widehat{\text{C}_r\text{MS}^h(\mathcal{R}, \mathcal{X})}$	normalized formulas	420
$F^{(n)}, t * \gamma$	representation of assignments	450
$L_{\varphi}, L_{\mathcal{C}, \varphi}^{\text{VR}}, L_{\mathcal{C}, \varphi}^{\text{HR}}, \mathcal{A}_{\varphi}, \mathcal{B}_{\mathcal{C}, \varphi}, \mathcal{C}_{\mathcal{C}, \varphi}$	sets of terms for models of $\varphi$ automata recognizing these sets	450, 460, 484 450, 461, 484
$\text{MINOR}_H$	sentence expressing minor inclusion	44

*Monadic second-order transductions (Chapters 7 and 9)*

$[\tau]_{\text{iso}}$	transduction of abstract structures	506
$\tau \cdot \tau', \tau \circ \tau'$	composition	506
$\tau^{-1}, \text{Dom}(\tau)$	inverse, domain	506
$\langle \chi, (\delta_i)_{i \in [k]}, (\theta_w)_{w \in \mathcal{R}' \otimes [k]} \rangle$	definition scheme	507
$\langle \chi, \delta, (\theta_R)_{R \in \mathcal{R}'} \rangle$	noncopying definition scheme	508
$qh(\mathcal{D})$	quantifier-height of $\mathcal{D}$	507
$\widehat{\mathcal{D}}$	transduction defined by $\mathcal{D}$	507
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witness, d2.43, d2.97, d3.55, s3.4.3

canonical, s2.3.2

ground, s2.3.2

simple, p6.45

word, d2.7

commutative, s3.1.6, s3.4.8

representation of, e5.2(1)