# 1 Euclidean geometry

*We familiarise ourselves with the axioms of Euclidean geometry in the plane and derive some geometric implications, among them the existence of the parallel line. We briefly discuss the historical importance of the parallel axiom. The existence of the Cartesian model shows the consistency of the Euclidean axioms. The Cartesian model is used to investigate Euclidean trigonometry.*

## 1.1 The axiomatic approach

Geometry is one of the oldest of all sciences. Remarkable geometric knowledge was already present in the advanced oriental cultures of the fifth–third centuries BCE. Practical problems from metrology, architecture, astronomy and navigation were considered on an abstract level and led to geometric laws. For instance, the Egyptians used the formula for the area of a triangle

$$\text{area} = \frac{\text{length of base line} \times \text{height}}{2}$$

and the approximate formula for the area of a circle

$$\text{area} = \left(\text{diameter} - \frac{\text{diameter}}{9}\right)^2.$$

The latter corresponds to an approximation of $\pi$ by $\frac{256}{81} \approx 3.1605$. No difference was made between exact and approximate formulae in principle. Mathematical knowledge was there in the form of laws, justifications or proofs were not given.

This changed in Greece between 350 and 200 BCE. Aspects of usefulness were then superseded by the desire for understanding. Mathematicians not only wanted to know certain laws, but also why they hold. This was the starting point of the axiomatisation of geometry. At first only a few intuitively evident axioms were laid down, from which it was thought that everything else could be derived logically in a rigorous manner. In what follows we will become

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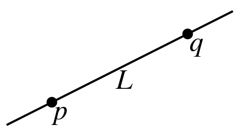
acquainted with the axioms of Euclidean plane geometry and go through some rather simple implications as illustrations of the axiomatic proof. We will mainly follow the formulation of the axioms presented by Hilbert in [13].

The axioms can be classified into five groups. We begin with the incidence axioms. To formulate these, we need two sets $\mathscr{P}$ and $\mathscr{G}$, whose elements we call points and straight lines respectively. Further assume that for every point $p \in \mathscr{P}$ and every straight line $L \in \mathscr{G}$ the statement "$p$ is contained in $L$", in symbols "$p \in L$", is either true or false. Note that the symbol "$\in$" does not denote a set-theoretic inclusion in this case, since the straight lines $L$ are for now not sets, but abstract elements of $\mathscr{G}$. We nevertheless want to use this suggestive notation. Let us now move on to the first axioms.

**Incidence axioms** These axioms make some statements about the containedness of points in straight lines.

Axiom I$_1$ *For any two points there exists a straight line that goes through both of them,*

$$\forall p, q \in \mathscr{P} \quad \exists L \in \mathscr{G}: \quad p \in L \text{ and } q \in L.$$




Axiom I$_2$ *There is at most one straight line through any two distinct points,*

$$\forall p, q \in \mathscr{P},\ p \neq q, \quad \forall L, M \in \mathscr{G},\ p \in L,\ q \in L,\ p \in M,\ q \in M: \quad L = M.$$

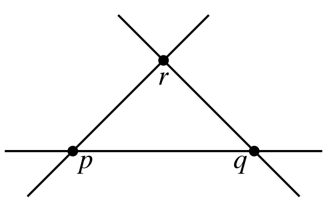
For any two distinct points $p$ and $q$, there is by those first two axioms exactly one straight line that goes through both of them, we will from now on denote it by $L(p, q)$.

Axiom I$_3$ *Every straight line contains at least two distinct points,*

$$\forall L \in \mathscr{G} \quad \exists p, q \in \mathscr{P},\ p \neq q: \quad p \in L \text{ and } q \in L.$$

Axiom I$_4$ *There exist three points that do not lie on a straight line,*

$$\exists p, q, r \in \mathscr{P}: \quad \nexists L \in \mathscr{G} \text{ with } p \in L,\ q \in L,\ r \in L.$$




Axiom I$_4$ expresses that our geometry has at least two dimensions.

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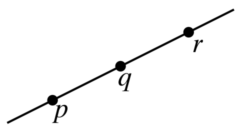
**Axioms of order** For the formulation of these axioms we need, in addition to the notions of $\mathscr{P}$, $\mathscr{G}$ and $\in$, that for every triple $(p, q, r)$ of points the statement "$q$ lies between $p$ and $r$" must be either true or false. The following axioms must be satisfied.

Axiom A1 *If $q$ lies between $p$ and $r$, then $p$, $q$ and $r$ are three pairwise distinct points on a straight line.*




Axiom A2 *If $q$ lies between $p$ and $r$, then $q$ lies between $r$ and $p$.*

For two points $p$ and $q$ we call the set of all points that lie between $p$ and $q$ the ***line segment*** from $p$ to $q$ and write $\overline{pq}$. Axiom A2 therefore implies $\overline{pr} = \overline{rp}$.

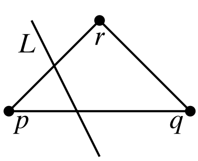
Axiom A3 *For any two distinct points $p$ and $q$ there exists a point $r$, such that $q$ lies between $p$ and $r$.*

*Attention* This axiom does not say that for any two given points, there exists another point between them. We will first have to *prove* this, see theorem 1.1.1.

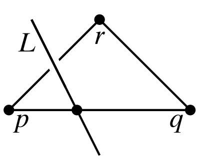
Axiom A4 *Given three points, at most one of them lies between the two others.*

If ***two straight lines*** $L$ and $M$ have a point $p$ in common, $p \in L$ and $p \in M$, then we sometimes say that $L$ and $M$ ***intersect***, in symbols $L \cap M \neq \emptyset$. We say that a ***line segment*** $\overline{pr}$ and a ***straight line*** $L$ ***intersect*** if there exists a point $q$ with $q \in L$ between $p$ and $r$.

Axiom A5 *Let $p$, $q$ and $r$ be three points that do not lie on a straight line, let $L$ be a straight line that does not contain any of these three points. If $L$ intersects the line segment $\overline{pq}$, then $L$ intersects precisely one of the other two line segments $\overline{pr}$ or $\overline{qr}$.*




This means that a straight line which enters a triangle must leave it through one of the other two sides. It also illustrates that our geometry does not have more than two dimensions. In three dimensions axiom A5 would not be valid:




Let us now prove a first theorem, using the axioms stated so far.

**Theorem 1.1.1** *For any two distinct points $p$ and $q$ there exists a point $r$ which lies between $p$ and $q$, i.e. the line segment $\overline{pq}$ is not empty.*

***Proof*** Let $p$ and $q$ be two points. By axiom $I_4$ there exists a point $s$ that does not lie on the straight line $L(p,q)$. By axiom $A_3$ there is a point $t$ such that $s$ lies between $p$ and $t$. Another application of axiom $A_3$ gives a point $u$ such that $q$ lies between $t$ and $u$. The straight line $L := L(s,u)$ intersects the line segment $\overline{pt}$ at $s$.

The point $t$ does not lie on the straight line $L(p,q)$, since otherwise $s$ would by axiom $A_1$ also lie on that straight line, contradicting the choice of $s$. We can therefore apply axiom $A_5$ to the straight line $L$ and the three points $p$, $q$ and $t$. As $L$ intersects the line segment $\overline{pt}$, $L$ must by axiom $A_5$ also intersect one of the two line segments $\overline{pq}$ or $\overline{tq}$, unless it contains one of the three points $p$, $q$ or $t$.

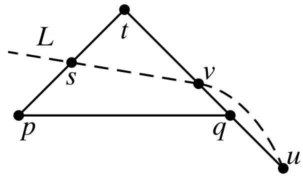
*First case* *$L$ contains $p$ or $t$.*

Then $L$ agrees with the straight line $L(p,t)$ by axiom $I_2$. Hence $u$ lies on $L(p,t)$ and axiom $A_1$ implies that $q$ lies on $L(p,t)$ as well. Hence $p$, $q$ and $t$ do lie on a straight line, a contradiction.

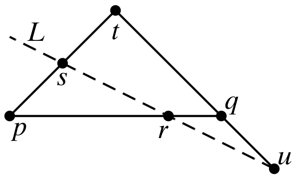
*Second case* *$L$ contains $q$.*

Several applications of axiom $I_2$ show that the points $s$, $u$, $q$, $t$ and $p$ lie on a straight line, a contradiction.

*Third case* *$L$ intersects the line segment $\overline{tq}$ at a point $v$.*




Then $L$ and $L(t,q)$ have the two points $u$ and $v$ in common. If $u = v$, then $u$ would lie between $t$ and $q$ while $q$ lies between $t$ and $u$, a contradiction to axiom $A_4$. Thus $L$ and $L(t,q)$ have two distinct points in common, and axiom $I_2$ implies that $L = L(t,q)$. But then both $s$ and $p$ lie on $L$, i.e. $p$, $q$ and $t$ lie on a straight line, a contradiction.




Thus $L$ and $\overline{pq}$ must intersect at a point $r$. In particular, $\overline{pq}$ cannot be empty. □

**Exercise 1.1** Let $p$, $q$, $r$ and $s$ be points on a straight line. Show that if $q$ lies between $p$ and $s$ and if additionally $r$ lies between $q$ and $s$, then $r$ lies between $p$ and $s$ as well.

Using this exercise and theorem 1.1.1, it is now easy to solve the following exercise.

**Exercise 1.2** Show the following: there is an infinite number of distinct points between two points.

**Definition 1.1.2** Let $L$ be a straight line, $p \in L$. Let $q$ and $r$ be two points on $L$, neither of which is $p$. We say that $q$ and $r$ ***lie on the same side of the point*** $p$ if $p$ does not lie between $q$ and $r$.

**Exercise 1.3** Let $L$ be a straight line and let $p \in L$ be a point on $L$. Show that the relation "$q_1$ lies on the same side of $p$ as $q_2$" defines an equivalence relation on the set $\{q \in L \mid q \neq p\}$.

An equivalence class of points on $L$ that do not equal $p$ can then be referred to as a ***side of*** $p$ ***on*** $L$.

**Exercise 1.4** Show that there are exactly two sides of $p$ on $L$.

**Definition 1.1.3** Let $L$ be a straight line and let $p$ and $q$ be two points that do not lie on $L$. We say that $p$ and $q$ ***lie on the same side of a straight line*** $L$ if the line segment $\overline{pq}$ does not intersect the straight line $L$.

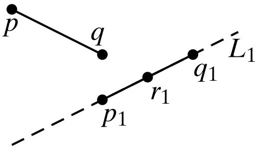
**Exercise 1.5** Let $L$ be a straight line. Show that the relation "$q_1$ lies on the same side of $L$ as $q_2$" defines an equivalence relation on the set $\{q \mid q \notin L\}$.

Again we can call an equivalence class of points not on $L$ a ***side of*** $L$.

**Exercise 1.6** Show that there are exactly two sides of $L$.

**Congruence axioms** To formulate the third group of axioms, the congruence axioms, we need in addition to the previous notions that for every pair $(\overline{pq}, \overline{p_1q_1})$ of line segments the formal statement "$\overline{pq}$ is congruent to $\overline{p_1q_1}$" is either true or false.

AXIOM $K_1$ (reproduction of lengths) *Let $\overline{pq}$ be a line segment, let $L_1$ be a straight line, let $p_1, r_1 \in L_1$, $r_1 \neq p_1$. Then there is a point $q_1 \in L_1$ on the same side of $p_1$ as $r_1$ such that $\overline{pq}$ is congruent to $\overline{p_1q_1}$.*



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In this axiom only the existence of a congruent line segment is required. Its uniqueness needs to be proved later with the aid of other axioms.

AXIOM $K_2$ *If the line segments $\overline{p_1q_1}$ and $\overline{p_2q_2}$ are both congruent to the line segment $\overline{pq}$, then $\overline{p_1q_1}$ is congruent to $\overline{p_2q_2}$ as well.*

Four more congruence axioms will follow. We can nevertheless already prove a first implication.

**Lemma 1.1.4** *The congruence of line segments defines an equivalence relation on the set of line segments.*

***Proof*** (a) Let $\overline{pq}$ be a line segment. We show that $\overline{pq}$ is congruent to itself. Let $L$ be a straight line that contains $p$, $p \in L$. By axiom $K_1$ there exists a point $r$ on $L$ such that $\overline{pq}$ and $\overline{pr}$ are congruent. Then $\overline{p_1q_1} := \overline{p_2q_2} := \overline{pq}$ is congruent to $\overline{pr}$ and hence $\overline{pq}$ is by axiom $K_2$ congruent to itself.

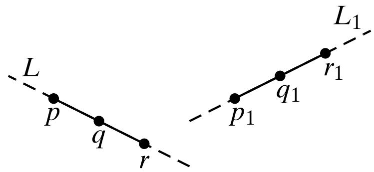
(b) (symmetry) Let $\overline{pq}$ be congruent to $\overline{p_1q_1}$. We show that then $\overline{p_1q_1}$ is congruent to $\overline{pq}$. It follows from (a) that $\overline{p_1q_1}$ is congruent to $\overline{p_1q_1}$. Axiom $K_2$ now gives that $\overline{p_1q_1}$ is congruent to $\overline{pq}$.

(c) (transitivity) If $\overline{p_1q_1}$ is congruent to $\overline{p_2q_2}$ and $\overline{p_2q_2}$ is congruent to $\overline{p_3q_3}$, then we need to show that $\overline{p_1q_1}$ is congruent to $\overline{p_3q_3}$. This follows directly from axiom $K_2$ together with (b). □

We will from now on sometimes denote "$\overline{p_1q_1}$ is congruent to $\overline{p_2q_2}$" by "$\overline{p_1q_1} \equiv \overline{p_2q_2}$".

AXIOM $K_3$ (additivity of line segments) *Let $L$ and $L_1$ be straight lines, let $p, q, r \in L$ be three pairwise distinct points on $L$ and $p_1, q_1, r_1 \in L_1$ likewise on $L_1$. Assume that the line segments $\overline{pq}$ and $\overline{qr}$ do not have any common points, $\overline{pq} \cap \overline{qr} = \emptyset$. Analogously let $\overline{p_1q_1} \cap \overline{q_1r_1} = \emptyset$.*

*If now $\overline{pq} \equiv \overline{p_1q_1}$ and $\overline{qr} \equiv \overline{q_1r_1}$, then $\overline{pr} \equiv \overline{p_1r_1}$.*




We need the concept of the *angle* for the formulation of the three other congruence axioms.

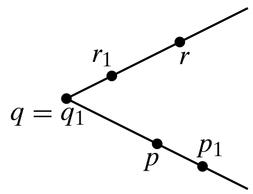
**Definition 1.1.5** An ***angle*** is an equivalence class of triples of points $p$, $q$ and $r$ that do not lie on a straight line, where two triples $(p, q, r)$ and $(p_1, q_1, r_1)$ are equivalent if

(i) $q = q_1$,
(ii) $L(p, q) = L(p_1, q)$ and $p$ and $p_1$ lie on the same side of $q$,
(iii) $L(r, q) = L(r_1, q)$ and $r$ and $r_1$ lie on the same side of $q$,

or if

(i) $q = q_1$,
(ii) $L(p, q) = L(r_1, q)$ and $p$ and $r_1$ lie on the same side of $q$,
(iii) $L(r, q) = L(p_1, q)$ and $r$ and $p_1$ lie on the same side of $q$.

For the equivalence class of $(p, q, r)$ we write $\angle(p, q, r)$. The point $q$ is then called the ***vertex*** of the angle $\angle(p, q, r)$.

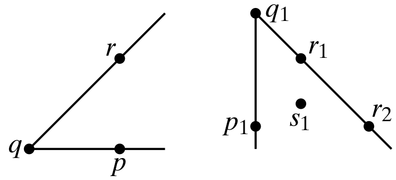



We now additionally require that for any two angles $\angle(p, q, r)$ and $\angle(p_1, q_1, r_1)$ the formal statement "$\angle(p, q, r)$ is congruent to $\angle(p_1, q_1, r_1)$" is either true or false. Again, we write "$\angle(p, q, r) \equiv \angle(p_1, q_1, r_1)$" if $\angle(p, q, r)$ is congruent to $\angle(p_1, q_1, r_1)$.

Axiom K4 *The congruence of angles induces an equivalence relation on the set of angles.*

Axiom K5 (reproduction of angles) *Let $p$, $q$, $r$ be points that do not lie on a straight line, and let $p_1$, $q_1$, $s_1$ be another set of points that do not lie on a straight line. Then there exists a point $r_1$ on the same side of $L(p_1, q_1)$ as $s_1$ such that the angle $\angle(p_1, q_1, r_1)$ is congruent to the angle $\angle(p, q, r)$.*

*If $r_2$ is another point with the same properties as $r_1$, i.e. $r_2$ also lies on the same side of $L(p_1, q_1)$ as $s_1$ and if $\angle(p_1, q_1, r_2) \equiv \angle(p, q, r)$, then $\angle(p_1, q_1, r_1) = \angle(p_1, q_1, r_2)$.*

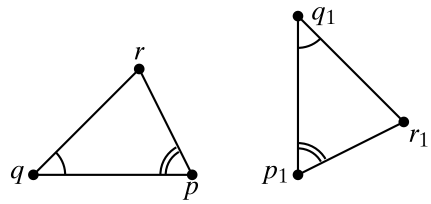



Axiom K5 says that we can reproduce a given angle in a unique way if we are given the vertex, one line adjacent to the angle and the side of the other line next to the angle.

The last congruence axiom relates the congruence of line segments to that of angles. Until now the two notions of congruence existed entirely separately.

AXIOM $K_6$ *Let $(p,q,r)$ be a triple of points that do not lie on a straight line and $(p_1,q_1,r_1)$ likewise. If $\overline{pq} \equiv \overline{p_1q_1}$, $\overline{pr} \equiv \overline{p_1r_1}$ and $\angle(q,p,r) \equiv \angle(q_1,p_1,r_1)$, then*

$$\angle(p,q,r) \equiv \angle(p_1,q_1,r_1).$$

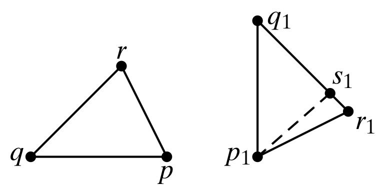



Let us make some inferences using the axioms introduced so far. We will first extend the statement of the last axiom.

**Theorem 1.1.6** *Let $(p,q,r)$ be a triple of points that do not lie on a straight line, $(p_1,q_1,r_1)$ likewise. If $\overline{pq} \equiv \overline{p_1q_1}$, $\overline{pr} \equiv \overline{p_1r_1}$ and $\angle(q,p,r) \equiv \angle(q_1,p_1,r_1)$, then*

$$\angle(p,q,r) \equiv \angle(p_1,q_1,r_1), \qquad \angle(p,r,q) \equiv \angle(p_1,r_1,q_1), \qquad \overline{qr} \equiv \overline{q_1r_1}.$$

***Proof*** The angle congruences follow directly from axiom $K_6$, in the second case after renaming the variables. It remains to show that $\overline{qr} \equiv \overline{q_1r_1}$. By axiom $K_1$ we can find a point $s_1$ on the straight line $L(q_1,r_1)$ which is on the same side as $r_1$ and satisfies $\overline{qr} \equiv \overline{q_1s_1}$.




We apply axiom $K_6$ to $(p,q,r)$ and $(p_1,q_1,s_1)$ and conclude that

$$\angle(q,p,r) \equiv \angle(q_1,p_1,s_1).$$

On the other hand we have $\angle(q,p,r) \equiv \angle(q_1,p_1,r_1)$, and by the uniqueness from axiom[1] $K_5$

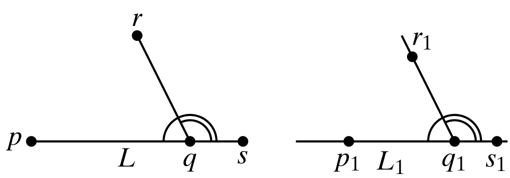
[1] For an application of the uniqueness statement from axiom $K_5$ it must be ensured that $r_1$ and $s_1$ lie on the same side of $L(p_1,q_1)$. This is left as an exercise for the reader.

$$\angle(q_1, p_1, s_1) = \angle(q_1, p_1, r_1).$$

If we now had $r_1 \neq s_1$, then we would conclude that $p_1$ and $q_1$ both lie on the straight line $L(r_1, s_1)$, i.e. $p_1$, $q_1$ and $r_1$ would lie on a straight line, which contradicts the assumption.

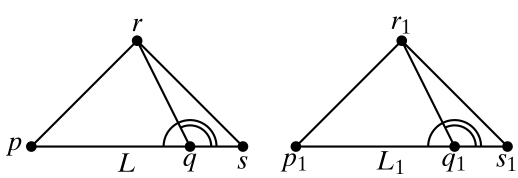
Thus $r_1 = s_1$ and hence $\overline{qr} \equiv \overline{q_1 r_1}$. □

**Theorem 1.1.7** (Congruence of adjacent angles) *Suppose that the pairwise distinct points $p$, $q$ and $s$ lie on a straight line $L$, while $r \notin L$. Analogously, let $p_1, q_1, s_1 \in L_1$ be pairwise distinct, $r_1 \notin L_1$. If $\angle(p,q,r)$ and $\angle(p_1,q_1,r_1)$ are congruent, then the same is true for $\angle(s,q,r)$ and $\angle(s_1,q_1,r_1)$.*




The angle $\angle(s,q,r)$ is sometimes called the ***adjacent angle*** of $\angle(p,q,r)$. The theorem thus says that adjacent angles of congruent angles are congruent as well.

***Proof*** The points $p_1$, $r_1$ and $s_1$ can by axiom $K_1$ be assumed to have been chosen in such a way that $\overline{pq} \equiv \overline{p_1 q_1}$, $\overline{rq} \equiv \overline{r_1 q_1}$ and $\overline{sq} \equiv \overline{s_1 q_1}$.




From theorem 1.1.6, applied to $(p,q,r)$ and $(p_1,q_1,r_1)$, it follows that $\overline{pr} \equiv \overline{p_1 r_1}$. By axiom $K_3$ we then have $\overline{ps} \equiv \overline{p_1 s_1}$. Applying theorem 1.1.6 once again, this time to $(r,p,s)$ and $(r_1,p_1,s_1)$, we obtain $\overline{rs} \equiv \overline{r_1 s_1}$ and $\angle(q,s,r) \equiv \angle(q_1,s_1,r_1)$. Axiom $K_6$ then says for $(q,s,r)$ and $(q_1,s_1,r_1)$ that $\angle(s,q,r) \equiv \angle(s_1,q_1,r_1)$. □

**Theorem 1.1.8** (Congruence of vertical angles) *Let $L$ and $M$ be two distinct straight lines that intersect at $p$. Let $r, q \in L$ lie on two distinct sides of $p$ and let $s, t \in M$ lie on two distinct sides of $p$ as well. Then*

$$\angle(q,p,s) \equiv \angle(r,p,t).$$

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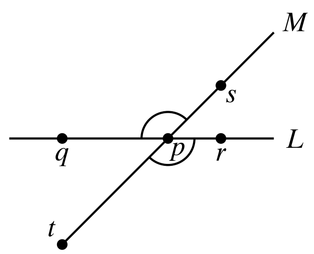
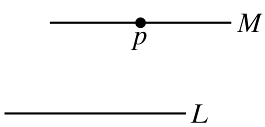



**Proof** Both $\angle(q,p,s)$ and $\angle(r,p,t)$, are adjacent angles of $\angle(q,p,t)$. The angle $\angle(q,p,t)$ is congruent to itself by axiom K4. The claim therefore follows by theorem 1.1.7. □

After these preparations we now come to the first truly interesting geometric theorem.

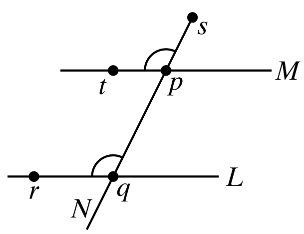
**Theorem 1.1.9** (Existence of a parallel) *Let $L$ be a straight line, $p$ a point, $p \notin L$. There then exists a straight line $M$ that contains $p$ and that does not intersect $L$.*




We then say that $M$ is a ***parallel*** to $L$ through $p$. The theorem says that such parallels always exist.

**Proof** Let $L$ be a straight line and $p$ a point that does not lie on $L$. We will first construct the line $M$ and then show that it has the desired properties.

For the construction we choose a point $q \in L$ and add the straight line $N := L(p,q)$. We choose another point $r \in L$, $r \neq q$. Then $r \notin N$, since otherwise $p \in L(p,q) = L(q,r) = L$. We reproduce angle $\angle(r,q,p)$ as in axiom K5 on the straight line $N$ at the point $p$, i.e. we find points $s \in N$ and $t \notin N$ on the same side of $N$ as $r$, such that the angle $\angle(t,p,s)$ is congruent to the angle $\angle(r,q,p)$. We now set $M := L(p,t)$.




It remains to show that $L$ and $M$ do not intersect. Suppose that $L$ and $M$ did intersect at a point $u$. We restrict ourselves to the case that $u$ lies on the same side of $N$ as $r$ and $t$. The other case is dealt with in a similar way.