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Basic concepts of dynamical systems theory

Everything should be made as simple as possible, but not simpler.
Albert Einstein

1.1 Deterministic systems

Since the Pythagorean attempts to explain the tangible world by means of numerical quantities related to integer numbers, western culture has been characterized by the idea that Nature can be described by mathematics. This idea comes from the explicit or hidden assumption that the world obeys some precise rules. It may appear obvious today, but the systematic application of mathematics to the study of natural phenomena dates from the seventeenth century when Galileo inaugurated modern physics with the publication of his major work *Discorsi e Dimostrazioni Matematiche Intorno a Due Nuove Scienze* (Discourses and Mathematical Demonstrations Concerning Two New Sciences) in 1638. The fundamental step toward the mathematical formalization of reality was taken by Newton and his mechanics, explained in *Philosophiae Naturalis Principia Mathematica* (The Mathematical Principles of Natural Philosophy), often referred to as the *Principia*, published in 1687. This was a very important date not only for the philosophy of physics but also for all the other sciences; this great work can be considered to represent the high point of the scientific revolution, in which science as we know it today was born. From the publication of the *Principia* to the twentieth century, for a large community of scientists the main goal of physics has been the reduction of natural phenomena to mechanical laws. A natural phenomenon was considered really understood only when it was explained in terms of mechanical movements.

The idea of determinism was established in a rather vivid way by Pierre Simon de Laplace (1814), in his book *Essai Philosophique sur les Probabilités* (Philosophical Essay on Probability):

We must consider the present state of Universe as the effect of its past state and the cause of its future state. An intelligence that would know all forces of nature and the respective situation of all its elements, if furthermore it was large enough to be able to analyze all these data, would embrace in the same expression the motions of the largest bodies of Universe as well as those of the slightest atom: nothing would be uncertain for this intelligence, all future and all past would be as known as present.

This statement has been a point of reference for scientific thought: a good scientific theory has to describe a natural phenomenon using mathematical methods. Once the temporal evolution equations of the phenomenon are written and the initial conditions are determined, the state of the system can be known at each future time by solving the equations. However, we would like to emphasize that Laplace was not naive at all about the true relevance of determinism (see later), as has sometimes been asserted by some writers of popular science.

1.1.1 Dynamical systems

Let us now introduce the notion of dynamical system. A deterministic dynamical system is essentially described by:

- (a) the phase space Ω , containing the vectors \mathbf{x} that determine, in a quantitative way, all the possible states of the system;
- (b) an evolution law $U(t, t_0)$, i.e. a rule that allows us to determine the state $\mathbf{x}(t)$ of the system at time t , given the state $\mathbf{x}(t_0)$ at time t_0 . Formally we can write

$$\mathbf{x}(t) = U(t, t_0)\mathbf{x}(t_0) = U(t - t_0)\mathbf{x}(t_0) \equiv U^{t-t_0}\mathbf{x}(t_0),$$

where, in the second equality, the stationarity of the evolution rule has been assumed, i.e. the system undergoes the same evolution from a given state \mathbf{x}_0 , independently from the time it is found in \mathbf{x}_0 . Moreover, U^t is a semigroup, that is $U^{r+s} = U^r U^s$ ($r, s > 0$) and $U^0 = I$, i.e. $\mathbf{x}(t_0) = U^0 \mathbf{x}(t_0)$.

The state of the system is typically specified by a d -dimensional vector \mathbf{x} , whose d components x_1, x_2, \dots, x_d , are called the degrees of freedom of the system. An elementary example is given by the pendulum, whose state is determined by the angle θ to the vertical and the angular velocity $\omega = d\theta/dt$; therefore the phase space is a cylindrical surface defined by $\theta \in [0, 2\pi]$ and $\omega \in [-\infty, +\infty]$: all the states of the pendulum are represented by points on this surface.

The most common deterministic evolution laws are maps and differential equations. In the first case the time is a discrete variable and the evolution law reads

$$\mathbf{x}(t+1) = \mathbf{g}[\mathbf{x}(t)] \quad (1.1)$$

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corresponding to the following system of d equations

$$\begin{aligned}x_1(t+1) &= g_1[x_1(t), x_2(t), \dots, x_d(t)] \\ \dots & \dots \dots \\ x_d(t+1) &= g_d[x_1(t), x_2(t), \dots, x_d(t)].\end{aligned}\tag{1.2}$$

In the case of differential equations the time is a continuous variable and the evolution law is prescribed as

$$\frac{d\mathbf{x}(t)}{dt} = \mathbf{f}[\mathbf{x}(t)]\tag{1.3}$$

which corresponds to the system of equations

$$\begin{aligned}\frac{d}{dt}x_1(t) &= f_1[x_1(t), x_2(t), \dots, x_d(t)] \\ \dots & \dots \dots \\ \frac{d}{dt}x_d(t) &= f_d[x_1(t), x_2(t), \dots, x_d(t)].\end{aligned}\tag{1.4}$$

The functions \mathbf{g} and \mathbf{f} in (1.1) and (1.3) do not contain an explicit time dependence, as a consequence of the stationarity assumption on the evolution. This assumption is not a severe limitation. A system can be made formally time independent by increasing by one unit the number of degrees of freedom.

The deterministic nature of the maps (1.1) is evident: given the initial state $\mathbf{x}(t_0)$, the state $\mathbf{x}(t)$ at time $t > t_0 = t - n$ is given by

$$\mathbf{x}(t) = \mathbf{g}[\mathbf{x}(t-1)] = \mathbf{g}[\mathbf{g}[\mathbf{x}(t-2)]] = \dots = \mathbf{g}^{(n)}[\mathbf{x}(t_0)],\tag{1.5}$$

where $\mathbf{g}^{(2)}(\mathbf{x}) = \mathbf{g}[\mathbf{g}[\mathbf{x}]]$, \dots , $\mathbf{g}^{(n)}(\mathbf{x}) = \mathbf{g}[\mathbf{g}^{(n-1)}[\mathbf{x}]]$.

The deterministic nature of the differential equations (1.3) is assured, under quite general conditions, by the existence and unicity theorem of the solution to a system of ordinary differential equations (Arnold 1974).

In particular, if $f_1(x_1, \dots, x_d), \dots, f_d(x_1, \dots, x_d)$ are linear functions of the variables x_1, \dots, x_d ,

$$\begin{aligned}f_1(x_1, \dots, x_d) &= a_{11}x_1 + a_{12}x_2 + \dots + a_{1d}x_d \\ \dots & \dots \dots \\ f_d(x_1, \dots, x_d) &= a_{d1}x_1 + a_{d2}x_2 + \dots + a_{dd}x_d,\end{aligned}\tag{1.6}$$

and if the a_{ij} coefficients are constant, the solution of the system can be easily written in an explicit form (Arnold 1974):

$$\mathbf{x}(t) = e^{At} \mathbf{x}(0),\tag{1.7}$$

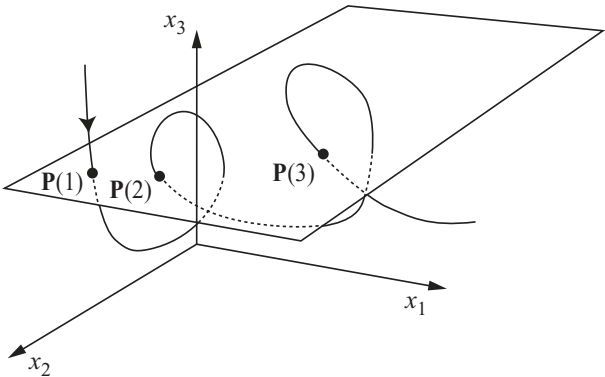


Figure 1.1 The generation of the Poincaré map by means of the Poincaré surface of section method in a three-dimensional flow.

where \mathbf{A} is the matrix whose elements are $\{a_{ij}\}$. An analogous result holds for linear map systems.

It is not difficult to understand that the maps and the differential equation systems are not completely disconnected representations of dynamical systems. For example, we can consider the simplest algorithm for the numerical integration of (1.4), i.e. the Euler scheme, to compute $\mathbf{x}(t + \tau)$ from $\mathbf{x}(t)$ with τ small enough: applying the definition of derivative, and neglecting terms of order τ^2 , one obtains the map

$$\begin{aligned} x_1(t + \tau) &= x_1(t) + f_1[x_1(t), x_2(t), \dots, x_d(t)]\tau \\ &\dots \quad \dots \quad \dots \\ x_d(t + \tau) &= x_d(t) + f_d[x_1(t), x_2(t), \dots, x_d(t)]\tau. \end{aligned} \tag{1.8}$$

Of course the Euler scheme is not very accurate. Nevertheless, more precise algorithms, for example the popular Runge–Kutta method, are nothing but maps which determine $\mathbf{x}(t + \tau)$ from $\mathbf{x}(t)$. Another way to reduce a continuous time dynamical system (or “flow”) to a discrete time map is through the Poincaré surface of section method. If we consider the d -dimensional flow (1.4), the *Poincaré map* gives its reduction to a $(d - 1)$ -dimensional map. For illustrative purposes, consider the three-dimensional case. The trajectory $\mathbf{x}(t)$ crosses the plane $x_3 = h$ with $dx_3/dt < 0$, the Poincaré surface, in the points $\mathbf{P}(0), \mathbf{P}(1), \dots, \mathbf{P}(n)$ at times t_0, t_1, \dots, t_n (see Figure 1.1). Since the point $\mathbf{x}(t_{n+1}) = (x_1(t_{n+1}), x_2(t_{n+1}), h)$ is determined uniquely by the point $\mathbf{x}(t_n)$, one has a deterministic rule connecting $\mathbf{P}(n)$ with $\mathbf{P}(n + 1)$, i.e.

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the Poincaré map which describes the evolution of the system on the plane:

$$\mathbf{P}(n+1) = \mathbf{g}[\mathbf{P}(n)]. \quad (1.9)$$

In general the explicit form of the Poincaré map associated with a given ordinary differential equation is not known, however its existence is useful for characterizing the behavior of the flow. For example, if the continuous time dynamical system is periodic, there will be only a finite number of isolated points on the Poincaré section. If the trajectory is quasi-periodic,¹ then there will be a regular closed figure, while if the trajectory is very irregular, there will be a non-structured set of points.

1.1.2 Attractors

The dynamical systems can be divided into two large classes: the *conservative* and the *dissipative* systems. A conservative dynamical system preserves the volume of the phase space. That is, given a region \mathcal{A}_0 , whose volume is V_0 , the points evolved from \mathcal{A}_0 define a region \mathcal{A}_t whose volume is $V_t = V_0$. This property is translated in differential terms as

$$\left| \det \left[\frac{\partial}{\partial x_i} g_j(\mathbf{x}) \right] \right| = 1 \quad \text{for maps,} \quad (1.10)$$

and

$$\nabla \cdot \mathbf{f} = \sum_{i=1}^d \frac{\partial}{\partial x_i} f_i(\mathbf{x}) = 0 \quad \text{for flows.} \quad (1.11)$$

An important example of a conservative system is given by Hamilton's equations for the motion of particles without friction. In contrast, a dissipative dynamical system does not preserve the volume of the phase space, i.e. $V_t < V_0$. The mathematical formulation of the contraction of the phase space in differential form is

$$\left| \det \left[\frac{\partial}{\partial x_i} g_j(\mathbf{x}) \right] \right| < 1 \quad \text{for maps,} \quad (1.12)$$

and

$$\nabla \cdot \mathbf{f} = \sum_{i=1}^d \frac{\partial}{\partial x_i} f_i(\mathbf{x}) < 0 \quad \text{for flows.} \quad (1.13)$$

¹ An N -frequency quasi-periodic motion can be represented by N independent variables, $f_1(t), f_2(t), \dots, f_N(t)$, such that each f_k is periodic with period T_k and the N frequencies $\Omega_i = 2\pi/T_i$ are incommensurate, that is, the relation $m_1\Omega_1 + m_2\Omega_2 + \dots + m_N\Omega_N = 0$ does not hold for any set of integers, m_1, m_2, \dots, m_N , except for the trivial solution $m_1 = m_2 = \dots = m_N = 0$. A two-frequency quasi-periodic motion lies on a two-dimensional torus.

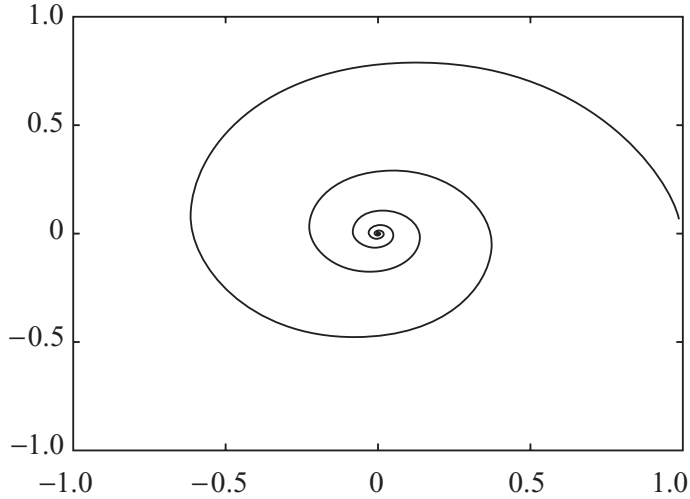


Figure 1.2 Example of a simple attractor: a stable fixed point.

A simple example of a dissipative system is the one-dimensional damped harmonic oscillator

$$\frac{d^2x}{dt^2} + \nu \frac{dx}{dt} + \omega^2 x = 0.$$

Because of the friction term $\nu dx/dt$, the system is dissipative and, as time goes on, the oscillation amplitude x and the velocity \dot{x} of the oscillator decrease and approach the asymptotic values $x = 0$, $\dot{x} = 0$. A trajectory in the phase space is shown in Figure 1.2 where the orbit spirals to the origin for any initial condition. In this case, the point $(0, 0)$ is an attracting point of the dynamical system.

Another example of a dissipative system is the pendulum clock, where the energy lost due to friction is reintegrated by a non-linear mechanism so that the oscillation amplitude is stabilized, as in the system described by the Van der Pol equation:

$$\frac{d^2x}{dt^2} + (x^2 - \nu) \frac{dx}{dt} + \omega^2 x = 0.$$

Figure 1.3 shows two typical trajectories of this kind of system: in both cases, the orbit, with time, spirals (inwards or outwards) to approach the closed curve on which it circulates in periodic motion in the $t \rightarrow \infty$ limit. The closed curve is a *limit cycle*.

As the above examples show, a very important property of dissipative systems is the presence of attracting sets or *attractors* in the phase space. These are bounded subsets of Ω to which regions of initial conditions of non-zero phase space volume

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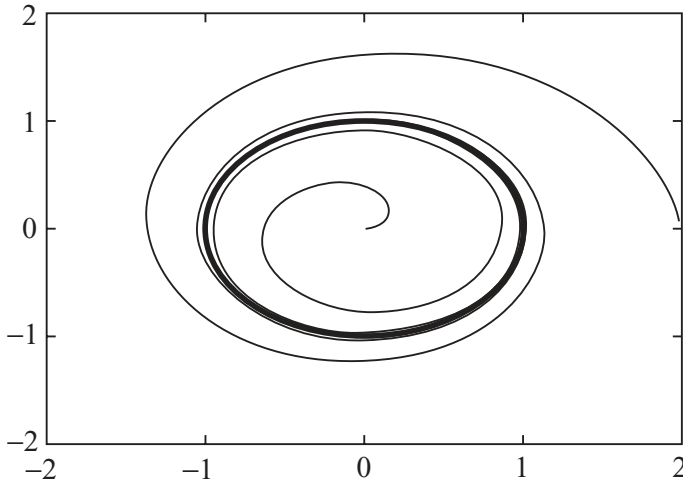


Figure 1.3 Example of a simple attractor: a stable limit cycle.

asymptote as time increases. From the property of volume preservation, it is easy to see that conservative dynamical systems do not possess attractors.

The attractors of the two continuous time systems considered above have a regular geometrical structure (a point, a closed curve) but this is not the case for all dissipative systems. An example of a non-trivial geometrical structure is the attractor of the two-dimensional Hénon map

$$\begin{cases} x(t+1) = 1 - ax^2(t) + y(t) \\ y(t+1) = bx(t). \end{cases} \quad (1.14)$$

Figure 1.4 shows the attractor of the Hénon map, for $a = 1.4$ and $b = 0.3$. The blow-up of the boxed region in Figure 1.4 (see Figure 1.5) reveals a small-scale pattern consisting of almost parallel lines. A further zoom in of a portion of Figure 1.5, shown in Figure 1.6, reveals that the part has the same structure as the whole. On continuing this zooming in procedure we would find a similar structure on arbitrarily small scales. This property of self-similarity qualifies the attractor as a *fractal*; see, e.g., Ott (1993). When the motions on the attractor, as in the case of the Hénon map, are also chaotic (see Section 1.3) the attractor is called a *strange attractor*.

1.2 Unpredictability: systems with many degrees of freedom

After Newton's foundation of the dynamical laws, the deterministic approach became a powerful and successful method for the understanding of natural phenomena especially in astronomy. As remarkable examples one can mention the derivation

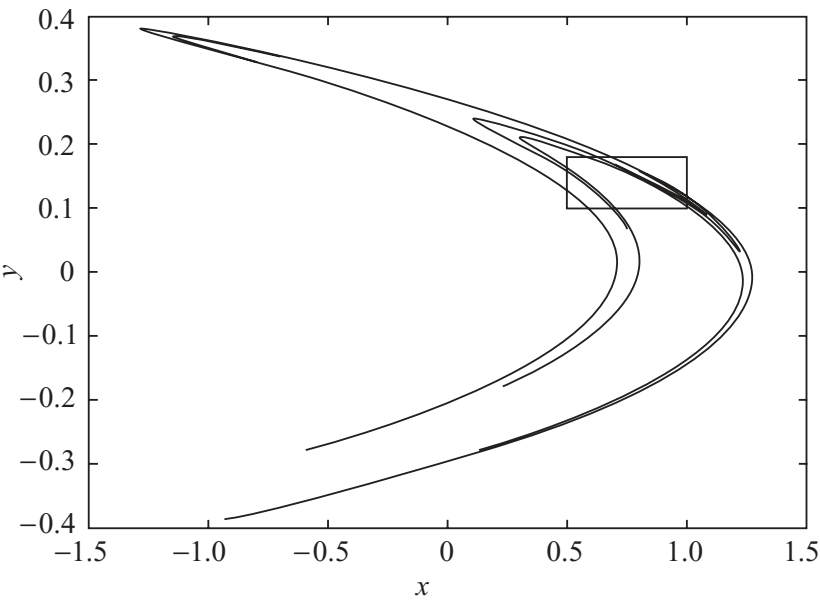


Figure 1.4 The attractor of the Hénon map, obtained using Eq. (1.14) with $a = 1.4$ and $b = 0.3$.

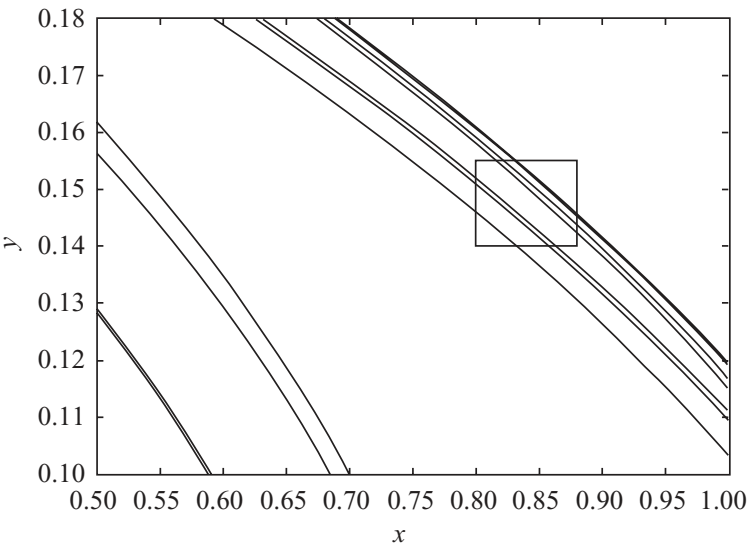


Figure 1.5 Enlargement of the boxed region in Figure 1.4.

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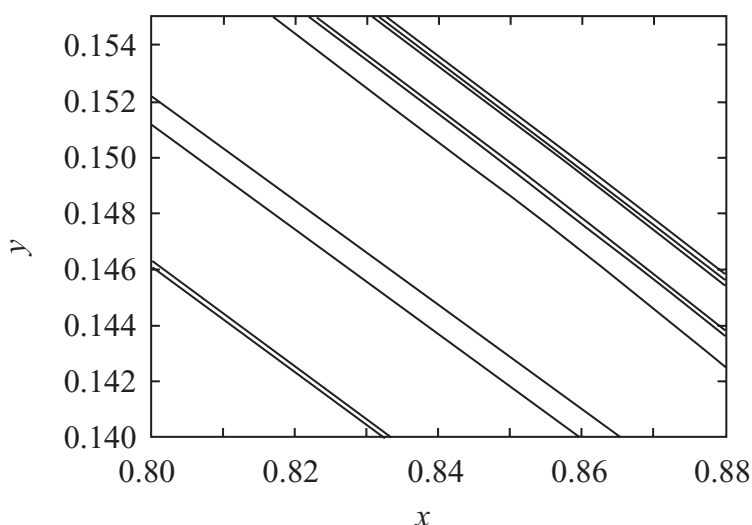


Figure 1.6 Enlargement of the boxed region in Figure 1.5.

of Kepler's laws from the Newtonian dynamical equations, and the gravitational force. Another paradigmatic success of Newtonian mechanics was the discovery of the planet Neptune, whose existence was predicted theoretically by Le Verrier and Adams. Today, the positions of many celestial bodies and artificial satellites can be calculated quickly with good accuracy by the powerful computers of astronomical study centers.

Nevertheless, everyday life is characterized by a lot of phenomena which exhibit unpredictable behaviors like the evolution of the weather or the fall of a leaf. How do we reconcile the deterministic Laplacian assumption with the “irregularity” and “unpredictability” of many natural phenomena? Laplace answered this question, again in his book *Essai Philosophique sur les Probabilités* (Philosophical Essay on Probability), by identifying the origin of the irregularity in our ignorance on the system:

The curve described by a simple molecule of air or vapor is regulated in a manner just as certain as the planetary orbits; the only difference between them is that which comes from our ignorance. Probability is relative, in part to this ignorance, in part to our knowledge.

Thus, according to the previous point of view, the observed irregularity is more apparent than real: it is due to a large number of simple reasons, for example, a large number of simple mechanical equations that rule the evolution of the system. This interpretation is at the basis of Langevin's approach to Brownian motion.

1.2.1 Brownian motion

In 1827 the Scottish botanist Robert Brown noticed that pollen grains suspended in water jiggled about under the lens of the microscope, following a zig-zag path. Initially, he believed that such activity was peculiar to the male sexual cells of plants, but then he observed that pollen of plants dead for over a century showed the same movement. Further study revealed that the same motion could be observed not only with particles of other organic substances but even with chips of glass or granite or particles of smoke.

In 1889 Gouy found that Brownian motion was more rapid for smaller particles, lower viscosity of the surrounding fluid and higher temperatures. These facts suggest that the basic cause of Brownian motion lies in the “thermal molecular motion in the liquid environment.” Therefore it is natural (at least today!), following the atomistic point of view, to suppose that a suspended particle is constantly and randomly bombarded from all sides by the molecules of the liquid.

After important and independent works by Einstein (1905) and Smoluchowski (1906), Langevin (1908) proposed an approach in terms of a stochastic differential equation (to use modern terminology) for the particle movement, taking into account the effect of the molecular hits by means of an average force, as given by the fluid friction, and a random fluctuating term.

The basic physical assumptions in both Einstein’s and Langevin’s approaches are

- (a) Stokes’s law for the friction of a body moving in a liquid;
- (b) equipartition of the kinetic energy among the various degrees of freedom of the system, i.e. between the particles of the fluid and the grain performing Brownian motion.

A colloidal particle suspended in a liquid at temperature T is somehow assimilated to a particle of the liquid, so that it possesses an average kinetic energy $RT/(2N_A)$, in each spatial direction, where R is the perfect gas constant and N_A is the Avogadro number (the number of molecules in one mole); therefore one has:

$$\frac{1}{2}m\langle v_x^2 \rangle = \frac{RT}{2N_A}. \quad (1.15)$$

According to Stokes’s law, a spherical particle of radius a , moving in a liquid with the speed v_x in the x direction experiences a viscous resistance:

$$F_{\text{Stokes}} = -\alpha v_x = -6\pi\eta a v_x,$$

where η is the viscosity. The above law holds if a is much larger than the average distance between the liquid molecules, and Stokes’s force represents the average macroscopic effect of the large number of irregular impacts of the molecules of the fluid.