INTEREST RATES AND COUPON BONDS IN QUANTUM FINANCE

The economic crisis of 2008 has shown that the capital markets need new theoretical and mathematical concepts to describe and price financial instruments.

Focusing on interest rates and coupon bonds, this book does not employ stochastic calculus – the bedrock of the present day mathematical finance – for any of the derivations. Instead, it analyzes interest rates and coupon bonds using quantum finance. The Heath–Jarrow–Morton model and the Libor Market Model are generalized by realizing the forward and Libor interest rates as an imperfectly correlated quantum field. Theoretical models have been calibrated and tested using bond and interest rates market data.

Building on the principles formulated in the author's previous book (*Quantum Finance*, Cambridge University Press, 2004), this ground-breaking book brings together a diverse collection of theoretical and mathematical interest rate models. It will interest physicists and mathematicians researching in finance, and professionals working in the finance industry.

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This book is dedicated to my wife Najma Sultana Baaquie,
my son Arzish Falaqul Baaquie, and my daughter Tazkiah Faizaan Baaquie.
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Prologue

The 2008 economic crisis has shown that the capital markets need new and fresh theoretical and mathematical concepts for designing and pricing financial instruments. Focusing on interest rates and coupon bonds, this book does not employ stochastic calculus – the bedrock of the present-day mathematical finance – for any of the derivations. Interest rates and coupon bonds are studied in the self-contained framework of quantum finance that is independent of stochastic calculus. Quantum finance provides solutions and results that go beyond the formalism of stochastic calculus.

It is five years since *Quantum Finance* [12] was published in 2004 and it is indeed gratifying to see how well it has been received. No attempt has been made to re-work the principles of finance. Rather, the main thrust of this book is to employ the methods of theoretical physics in addressing the subject of finance. Theoretical physics has accumulated a vast and rich repertoire of mathematical concepts and techniques; it is only natural that this treasure house of quantitative tools be employed to analyze the field of finance, and the debt market in particular.

The term 'quantum' in *Quantum Finance* refers to the use of *quantum mathematics*, namely the mathematics and theoretical concepts of quantum mechanics and quantum field theory, in analyzing and studying finance. Finance is an entirely classical subject and there is no $\hbar$ – Planck's constant, the *sine qua non* of quantum phenomena – in quantum finance: the term 'quantum' is a *metaphor*. Consider the case of classical phase transformations that result from the random fluctuations of classical fields; critical exponents, which characterize phase transitions, are computed using the mathematics of nonlinear quantum field theories [95]. Similar to the case of phase transitions, quantum mathematics provides powerful theoretical and mathematical tools for studying the underlying random processes that drive modern finance.

The principles of quantum finance provide a comprehensive and self-contained theoretical platform for modeling all forms of financial instruments. This book,

in particular, is focused on studying interest rates and coupon bonds. A detailed analytical, computational, and empirical study of debt instruments constitutes the main content of this book.

The Libor Market Model and the Heath–Jarrow–Morton model, which are the industry standards for modeling interest rates and coupon bonds, are both based on exactly correlated Libor and forward interest rates. The book makes a quantum finance generalization of these models to *imperfectly correlated* interest rates by modeling the forward interest rates as a quantum field. Empirical studies provide strong evidence supporting the imperfect correlation of interest rates. Many groundbreaking results are obtained for debt instruments. In particular, it is shown that quantum field theory provides a generalization of Ito calculus that is required for studying imperfectly correlated interest rates.

In the capital markets, interest rates determine the returns on cash deposits. Coupon bonds, on the other hand, are loans that are disbursed – with the objective of earning interest – against promissory notes. In principle, the interest paid on cash deposits and the interest earned on loans are equivalent. However, all interest rates are only defined for a *finite* time interval – of which the minimum is overnight (24 hours). In particular, all interest rate *derivatives* are based on benchmark interest rates for cash deposits of a duration of 90 days. The bond (derivatives) markets, in contrast, have no such minimum duration. The existence of a finite duration for the (benchmark) interest rates creates *two distinct sectors* of the debt derivatives market, namely derivatives of interest rates and derivatives of coupon bonds – with a nonlinear transformation connecting the two sectors.

Numerous and exhaustive calculations are carried out for diverse forms of interest rate and coupon bond options. Complicated concepts and calculations that are typical for debt instruments are introduced and motivated, in some cases by first discussing analogous and simpler equity instruments. It is my view that only by actually working out the various steps required in a calculation can a reader grasp the principles and techniques of what is still a subject in its infancy. Almost all the intermediate steps in the various calculations are included so as to clear the way for the interested reader. A few key ideas are repeated in the various chapters so that each chapter can be read more or less independently.

The material covered in the book is primarily meant for physicists and mathematicians engaged with research in the field of finance, as well as professional theorists working in the finance industry. Specialists working in the field of debt instruments will hopefully find that the theoretical tools and mathematical ideas developed in this book broaden their repertoire of quantitative approaches to finance. The material could also be of interest to physicists, probabilists, applied mathematicians, and statisticians – as well as graduates students in science and engineering – who are thinking of pursuing research in the field of finance.

One of the aims of this book is to be self-contained and comprehensive. All derivations and concepts are introduced from first principles, and all important results are derived *ab initio*. Given the diverse nature of the potential audience, fundamental concepts of finance have been reviewed for readers who are new to this field. Appendix A reviews the essential mathematical background required for following the various derivations and is meant to introduce specialists working in finance to the concepts of quantum mathematics.

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