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Introduction

After Einstein first presented his theory of general relativity in 1915, a few exact solutions of his field equations were found very quickly. All of these assumed a high degree of symmetry. Some could be interpreted as representing physically significant situations such as the exterior field of a spherical star, or a homogeneous and isotropic universe, or plane or cylindrical gravitational waves. Yet it took a long time before some of the more subtle properties of these solutions were widely understood.

In their seminal review of "exact solutions of the gravitational field equations", Ehlers and Kundt (1962) included the following statement. "At present the main problem concerning solutions, in our opinion, is not to construct more but rather to understand more completely the known solutions with respect to their local geometry, symmetries, singularities, sources, extensions, completeness, topology, and stability." Since this was written, considerable progress has been made in the understanding of many exact solutions. However, this development has been very restricted compared to the enormous effort that has been put into the derivation of further "new" solutions. Although significant advance has been achieved in the interpretation of many solutions, it is a fact that some aspects of even the most frequently quoted exact solutions still remain poorly understood. The opinion of Ehlers and Kundt thus still indicates an even more urgent task.

In this work, the very traditional approach will be adopted that an exact solution of Einstein's equations is expressed in terms of a metric in particular coordinates. Specifically, it will be represented in the form of a 3+1-dimensional line element in which the coordinates have certain ranges. Our purpose will be to try to identify its physical interpretation. Does it represent a specific physical situation? Does it include singularities or horizons, and what do these mean? For what range of the coordinates is the solution valid, or can it be extended by the introduction of different coordinates? How does it behave asymptotically? Does it approach other known solutions in specific limits? These and other questions will be used to probe the meaning of the solutions considered.

It must be remembered of course, that an exact solution of Einstein's equations as defined above is initially just a *local* solution of the field equations. Global and topological properties of the associated manifold may be chosen according to our own prejudices. They are not implied by the field equations. Thus it may well occur that some particular exact solution may have a number of very different physical interpretations. For example, part of the well-known Schwarzschild solution may either represent part of the space-time inside the horizon of a black hole, or it could represent the interaction region following the collision of two specific plane gravitational waves.

Such cases in which a particular solution has a number of possible interpretations are, however, unusual. It is far more likely that a solution has no useful physical significance at all. Nevertheless, each space-time may at least be understood in terms of its geometrical properties. And it could well be that realistic physical situations may be approximated by compound space-times formed by patching different local exact solutions in appropriate ways. To construct such space-times, it is necessary to have some understanding of the properties of each component. Thus, although this work will concentrate on the simpler solutions that have clear physical meanings, we will also describe the basic properties of related solutions even when their immediate applicability is unclear.

Einstein's general theory of relativity is a covariant theory. The same physical space-time may be expressed (at least locally) in any number of different coordinate representations. Smooth coordinate transformations can be applied without changing the character of the physical space-time itself. However, for any particular space-time, some coordinate systems are more useful than others. Some may be convenient because they enable the field equations to be expressed in forms that have nice mathematical properties. Others may be more useful for a physical interpretation of the space-time. However, when transforming from one coordinate system to another, the different coordinates may not be in a simple one-to-one correspondence with each other over their natural ranges. In such cases, the different coordinates may span different portions of the complete space-time. Care therefore has to be taken in specifying the ranges of the coordinates employed, and also in identifying whether or not the boundaries of the coordinate patch correspond to the boundaries of the physical space-time being represented.

This principle of general covariance also has significant implications in the derivation of "new" solutions of the field equations. It is generally unclear initially whether a solution that is newly obtained is a known solution in an unfamiliar coordinate system or represents a previously unknown space-time. This is referred to as "the equivalence problem" that has now been widely addressed in the literature (see for example Chapter 9 in Stephani *et al.*, 2003).

In presenting a solution of Einstein's equations, it is now standard practice to identify its local sources (i.e. the structure of the Einstein tensor, and hence of the energy-momentum tensor), the algebraic type of the Weyl tensor, its curvature invariants, and the number and type of its symmetries. These are all essential in assisting to classify the solution, and hence to determine whether or not it is genuinely new rather than a new coordinate representation of some previously known solution. These properties are also important for its physical interpretation.

The solutions described in this work mostly represent vacuum space-times. Often a cosmological constant or an electromagnetic field will be included, or occasionally even a pure radiation field. However, we have severely restricted the number of solutions included that have a perfect fluid source. The reason for this decision is simply that, although such solutions are often interpreted as possible cosmological models or stellar interiors, they are already thoroughly reviewed in published literature. In particular, we would recommend the classic text of Ryan and Shepley (1975) on homogeneous cosmological models and the excellent and complementary book of Krasiński (1997) on inhomogeneous models.

We initially deal with the fundamental Minkowski, de Sitter and anti-de Sitter spaces and the Friedmann–Lemaître–Robertson–Walker universes, which are all conformally flat and highly symmetric. We then address solutions which have Weyl tensors that are of the special algebraic types D and N, which generally represent the simplest non-radiating and radiating solutions. And we finally proceed to address some algebraically more general solutions. As some of the simplest known solutions of Einstein's equations, most of those described here have a high degree of symmetry. However, we will not generally identify all the Killing vectors, and the existence of homothetic and conformal symmetries will be largely ignored.

A basic knowledge of Einstein's theory of relativity is assumed throughout this work. On the other hand, since we will not describe any method by which exact solutions are obtained, it will not be necessary to introduce much unfamiliar notation or advanced techniques. We trust that the notation used will be familiar, modulo certain sign conventions.

For the specific notation employed in this book, we have tried to follow as closely as appropriate that of the "exact solutions" book of Stephani *et al.* (2003). This also includes some very helpful introductory surveys of certain important topics in general relativity and differential geometry that therefore need not be repeated here.

We very willingly acknowledge that there have been many previous reviews of various exact solutions of Einstein's equations which emphasise aspects of their physical interpretation. We are therefore building on a sure foundation. However, the subject has now become so vast that any review must be selective and will inevitably reflect the prejudices of the authors. A balance also has to be struck between pedagogy and a review of current research. Each review has been addressed to a particular need, and we trust that our present contribution to the literature will be sufficiently different as to be considered a welcome addition.

Published work on the particular topics discussed in this book will be cited in the relevant sections. However, some general reviews apply more widely and are more appropriately cited here.

An early review by Ehlers and Kundt (1962) has had a significant impact on all later work. Understanding of the global properties of space-times was greatly advanced by the book of Hawking and Ellis (1973). Another seminal work that has had a major impact on the subject is the "exact solutions" book of Kramer *et al.* (1980). A most welcome second edition of this is now available (Stephani *et al.*, 2003). This provides an exhaustive review of known solutions at that time, but does not usually emphasise their physical interpretation.

Reviews of exact solutions with an emphasis on their interpretation have been given by Bonnor (1982, 1992), Bonnor, Griffiths and MacCallum (1994) and Bičák (2000a). Reviews of general families of radiative space-times have been given by Bičák (1989, 1997, 2000b) and Bičák and Krtouš (2003). For a brief and modern introduction to this subject, see Bičák (2006).

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Basic tools and concepts

The purpose of this chapter is to define the notation that is used in this book, to introduce some basic tools that are employed, and to make a few initial comments on some of the concepts involved. It is not intended as a review of the topics mentioned, as these are described thoroughly in existing textbooks on general relativity.

2.1 Local geometry

Throughout this book, a solution of Einstein's equations is assumed to be given in terms of a metric, that is expressed in some local coordinate system, and which could represent a particular region of some theoretically possible space-time.

Space-time is assumed to be 3+1-dimensional. Taking the timelike coordinate first, the metric is assumed to have a Lorentzian signature $(-,+,+,+)$ so that timelike vectors have negative magnitude. It is represented (at least locally) by a manifold $\mathcal{M}$ with a symmetric metric $\mathbf{g}$ with coordinate components $g_{\mu\nu}$, where Greek letters span 0,1,2,3. The inverse of $g_{\mu\nu}$ is denoted by $g^{\mu\nu}$.

The speed of light is taken to be unity so that time and distance are measured by the same (unspecified) units, and null cones in space-time diagrams are normally drawn at an angle of 45° to the vertical.

The manifold $\mathcal{M}$ is assumed to be endowed with a linear (metric) connection that can be expressed in a coordinate basis in the form

$$\Gamma^{\lambda}{}_{\mu\nu} = \tfrac{1}{2}\, g^{\lambda\alpha}(g_{\mu\alpha,\nu} + g_{\nu\alpha,\mu} - g_{\mu\nu,\alpha}),$$

where the summation convention is adopted and a comma denotes a partial derivative. A semi-colon is used to denote a covariant derivative, so that the covariant derivative of a vector V^μ is given by

$$V^\mu{}_{;\nu} = V^\mu{}_{,\nu} + \Gamma^\mu{}_{\nu\alpha} V^\alpha.$$

It is frequently important, at any event (or point in space-time), to determine the components of vectors or tensors in particular directions. For this, it is first appropriate to introduce a normalised orthonormal tetrad $\boldsymbol{t}, \boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z}$, composed of a timelike and three spacelike vectors. From these, it is convenient to construct a null tetrad $\boldsymbol{k}, \boldsymbol{l}, \boldsymbol{m}, \bar{\boldsymbol{m}}$, with the two null vectors $\boldsymbol{k} = \frac{1}{\sqrt{2}}(\boldsymbol{t}+\boldsymbol{z})$ and $\boldsymbol{l} = \frac{1}{\sqrt{2}}(\boldsymbol{t}-\boldsymbol{z})$, and the complex vector $\boldsymbol{m} = \frac{1}{\sqrt{2}}(\boldsymbol{x} - \mathrm{i}\,\boldsymbol{y})$ and its conjugate $\bar{\boldsymbol{m}} = \frac{1}{\sqrt{2}}(\boldsymbol{x} + \mathrm{i}\,\boldsymbol{y})$, which span the 2-spaces orthogonal to $\boldsymbol{k}$ and $\boldsymbol{l}$. These null tetrad vectors are mutually orthogonal except that $k_\mu\, l^\mu = -1$ and $m_\mu\, \bar{m}^\mu = 1$. With these conditions, the metric tensor can be expressed in terms of its null tetrad components in the form

$$g_{\mu\nu} = -k_\mu \ell_\nu - \ell_\mu k_\nu + m_\mu \bar{m}_\nu + \bar{m}_\mu m_\nu.$$

Such a null tetrad may be transformed in the following ways:

$$\boldsymbol{k}' = \boldsymbol{k}, \qquad \boldsymbol{l}' = \boldsymbol{l} + L\,\bar{\boldsymbol{m}} + \bar{L}\,\boldsymbol{m} + L\bar{L}\,\boldsymbol{k}, \qquad \boldsymbol{m}' = \boldsymbol{m} + L\,\boldsymbol{k}, \tag{2.1}$$

$$\boldsymbol{k}' = \boldsymbol{k} + K\,\bar{\boldsymbol{m}} + \bar{K}\,\boldsymbol{m} + K\bar{K}\,\boldsymbol{l}, \qquad \boldsymbol{l}' = \boldsymbol{l}, \qquad \boldsymbol{m}' = \boldsymbol{m} + K\,\boldsymbol{l}, \tag{2.2}$$

$$\boldsymbol{k}' = B\,\boldsymbol{k}, \qquad \boldsymbol{l}' = B^{-1}\boldsymbol{l}, \qquad \boldsymbol{m}' = \mathrm{e}^{\mathrm{i}\Phi}\boldsymbol{m}, \tag{2.3}$$

where L and K are complex and B and Φ are real parameters. Together, these represent the six-parameter group of Lorentz transformations.

2.1.1 Curvature

Using the above notation, the (Riemann) curvature tensor, defined such that

$$V^\kappa{}_{;\mu\nu} - V^\kappa{}_{;\nu\mu} = -R^\kappa{}_{\lambda\mu\nu} V^\lambda,$$

is given by

$$R^\kappa{}_{\lambda\mu\nu} = \Gamma^\kappa{}_{\lambda\nu,\mu} - \Gamma^\kappa{}_{\lambda\mu,\nu} + \Gamma^\alpha{}_{\lambda\nu}\Gamma^\kappa{}_{\alpha\mu} - \Gamma^\alpha{}_{\lambda\mu}\Gamma^\kappa{}_{\alpha\nu}. \tag{2.4}$$

This generally has 20 independent components according to the symmetries $R^\kappa{}_{\lambda(\mu\nu)} = 0$, $R^\kappa{}_{[\lambda\mu\nu]} = 0$ and $R_{(\kappa\lambda)\mu\nu} = 0$, where round and square brackets are used to denote the symmetric and antisymmetric parts, respectively. Defining the (symmetric) Ricci tensor and the Ricci scalar by

$$R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}, \qquad R = g^{\alpha\beta} R_{\alpha\beta}, \tag{2.5}$$

the trace-free part of the curvature tensor is given explicitly by

$$
\begin{aligned}
C_{\kappa\lambda\mu\nu} = R_{\kappa\lambda\mu\nu} + \tfrac{1}{2}(R_{\lambda\mu}g_{\kappa\nu} + R_{\kappa\nu}g_{\lambda\mu} - R_{\lambda\nu}g_{\kappa\mu} - R_{\kappa\mu}g_{\lambda\nu}) \\
+\tfrac{1}{6}R\,(g_{\kappa\mu}g_{\lambda\nu} - g_{\kappa\nu}g_{\lambda\mu}).
\end{aligned}
\tag{2.6}
$$

This is known as the Weyl tensor which, in general, has ten independent components.

The curvature tensor can be expressed in terms of its various tetrad components. In particular, the ten independent components of the Ricci tensor are determined by the scalar quantities defined as[1]

$$
\begin{aligned}
\Phi_{00} &= \tfrac{1}{2}\,R_{\mu\nu}\,k^\mu\,k^\nu, & \Phi_{22} &= \tfrac{1}{2}\,R_{\mu\nu}\,l^\mu\,l^\nu, \\
\Phi_{01} &= \tfrac{1}{2}\,R_{\mu\nu}\,k^\mu\,m^\nu, & \Phi_{12} &= \tfrac{1}{2}\,R_{\mu\nu}\,l^\mu\,m^\nu, \\
\Phi_{02} &= \tfrac{1}{2}\,R_{\mu\nu}\,m^\mu\,m^\nu, & \Phi_{11} &= \tfrac{1}{4}\,R_{\mu\nu}(k^\mu\,l^\nu + m^\mu\,\bar{m}^\nu),
\end{aligned}
\tag{2.7}
$$

in which Φ_{AB} are generally complex but satisfy the constraint $\bar{\Phi}_{AB} = \Phi_{BA}$, and the Ricci scalar R.

The ten independent components of the Weyl tensor are similarly determined by the five complex scalar functions defined as

$$
\begin{aligned}
\Psi_0 &= C_{\kappa\lambda\mu\nu}\,k^\kappa\,m^\lambda\,k^\mu\,m^\nu, \\
\Psi_1 &= C_{\kappa\lambda\mu\nu}\,k^\kappa\,l^\lambda\,k^\mu\,m^\nu, \\
\Psi_2 &= C_{\kappa\lambda\mu\nu}\,k^\kappa\,m^\lambda\,\bar{m}^\mu\,l^\nu, \\
\Psi_3 &= C_{\kappa\lambda\mu\nu}\,l^\kappa\,k^\lambda\,l^\mu\,\bar{m}^\nu, \\
\Psi_4 &= C_{\kappa\lambda\mu\nu}\,l^\kappa\,\bar{m}^\lambda\,l^\mu\,\bar{m}^\nu.
\end{aligned}
\tag{2.8}
$$

By considering the equation of geodesic deviation (see below) in a suitably adapted frame, these components (in vacuum space-times) may be shown generally to have the following physical meanings:

$$
\begin{aligned}
&\Psi_0 \text{ is a transverse component propagating in the } \boldsymbol{l} \text{ direction,} \\
&\Psi_1 \text{ is a longitudinal component in the } \boldsymbol{l} \text{ direction,} \\
&\Psi_2 \text{ is a Coulomb-like component,} \\
&\Psi_3 \text{ is a longitudinal component in the } \boldsymbol{k} \text{ direction,} \\
&\Psi_4 \text{ is a transverse component propagating in the } \boldsymbol{k} \text{ direction.}
\end{aligned}
\tag{2.9}
$$

According to Einstein's general theory of relativity, the curvature of space-time is related to the distribution of matter. Specifically, components of the

[1] The notation for these components, and those of the Weyl tensor given below, is closely related to that of Newman and Penrose (1962). However, a variant of the Newman–Penrose formalism is required here because a different signature is employed. The notation used here is that given in Stephani *et al.* (2003).

Ricci tensor are directly related to the local energy-momentum tensor $T_{\mu\nu}$ by Einstein's field equations

$$R_{\mu\nu} - \tfrac{1}{2}\,R\,g_{\mu\nu} + \Lambda\,g_{\mu\nu} = 8\pi\,T_{\mu\nu}, \tag{2.10}$$

in which units of mass have been chosen so that $G=1$, and Λ is the *cosmological constant*. This can also be rewritten in terms of the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\,R\,g_{\mu\nu}$.

The trace-free part of the curvature tensor (i.e. the Weyl tensor), however, is determined only indirectly from the field equations. These components may therefore be understood as representing "free components" of the gravitational field that also arise from non-local sources. In seeking to interpret any exact solution physically, these components need to be investigated explicitly.

2.1.2 Algebraic classification

For reasons that will be clarified in Section 2.3.3, a space-time is said to be *conformally flat* if its Weyl tensor vanishes, i.e. if $C_{\kappa\lambda\mu\nu}=0$. Otherwise, gravitational fields are usually classified according to the Petrov–Penrose classification of their Weyl tensor. This is based on the number of its distinct principal null directions and the number of times these are repeated.

A null vector $\boldsymbol{k}$ is said to be a *principal null direction* of the gravitational field if it satisfies the property

$$k_{[\rho}C_{\kappa]\lambda\mu[\nu}k_{\sigma]}k^{\lambda}k^{\mu} = 0. \tag{2.11}$$

If $\boldsymbol{k}$ is a member of the null tetrad defined above, then the condition (2.11) is equivalent to the statement that $\Psi_0=0$. It may then be noted that, under a transformation (2.2) of the tetrad which keeps $\boldsymbol{l}$ fixed, but changes the direction of $\boldsymbol{k}$, the component Ψ_0 of the Weyl tensor transforms as

$$\Psi_0 = {\Psi_0}' - 4K{\Psi_1}' + 6K^2{\Psi_2}' - 4K^3{\Psi_3}' + K^4{\Psi_4}'.$$

The condition for $\boldsymbol{k}$ to be a principal null direction, i.e. that $\Psi_0=0$, is thus equivalent to the existence of a root K such that

$${\Psi_0}' - 4K{\Psi_1}' + 6K^2{\Psi_2}' - 4K^3{\Psi_3}' + K^4{\Psi_4}' = 0. \tag{2.12}$$

Since this is a quartic expression in K, there are four (complex) roots to this equation, although these do not need to be distinct.

Each root of (2.12) corresponds to a principal null direction which can be constructed using (2.2), and the multiplicity of each principal null direction is the same as the multiplicity of the corresponding root. For a principal null direction $\boldsymbol{k}$ of multiplicity 1, 2, 3 or 4, it can be shown that, respectively

$$\begin{array}{rcll} k_{[\rho}C_{\kappa]\lambda\mu[\nu}k_{\sigma]}k^\lambda k^\mu = 0 & \Leftrightarrow & \Psi_0 = 0, & \Psi_1 \neq 0, \\ C_{\kappa\lambda\mu[\nu}k_{\sigma]}k^\lambda k^\mu = 0 & \Leftrightarrow & \Psi_0 = \Psi_1 = 0, & \Psi_2 \neq 0, \\ C_{\kappa\lambda\mu[\nu}k_{\sigma]}k^\mu = 0 & \Leftrightarrow & \Psi_0 = \Psi_1 = \Psi_2 = 0, & \Psi_3 \neq 0, \\ C_{\kappa\lambda\mu\nu}k^\mu = 0 & \Leftrightarrow & \Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0, & \Psi_4 \neq 0. \end{array}$$

If a space-time admits four distinct principal null directions (*pnd*s), it is said to be *algebraically general*, or of type I, otherwise it is *algebraically special*. The distinct algebraic types can be summarised as follows:

type I	:	four distinct *pnd*s
type II	:	one *pnd* of multiplicity 2, others distinct
type D	:	two distinct *pnd*s of multiplicity 2
type III	:	one *pnd* of multiplicity 3, other distinct
type N	:	one *pnd* of multiplicity 4
type O	:	conformally flat

If either of the basis vectors $\boldsymbol{k}$ or $\boldsymbol{l}$ are aligned with principal null directions, either $\Psi_0 = 0$ or $\Psi_4 = 0$, respectively. If the vector $\boldsymbol{k}$ is aligned with the repeated principal null direction of an algebraically special space-time, then $\Psi_0 = 0 = \Psi_1$. If $\boldsymbol{k}$ and $\boldsymbol{l}$ are both aligned with the two repeated principal null directions of a type D space-time, then the only non-zero component of the Weyl tensor is Ψ_2. For a type N space-time with repeated principal null direction $\boldsymbol{k}$, the only non-zero component of the Weyl tensor is Ψ_4.

Two particularly useful complex scalar polynomial invariants for a vacuum space-time are given in terms of the Weyl tensor components by

$$I = \Psi_0\Psi_4 - 4\,\Psi_1\Psi_3 + 3\,{\Psi_2}^2, \qquad J = \begin{vmatrix} \Psi_0 & \Psi_1 & \Psi_2 \\ \Psi_1 & \Psi_2 & \Psi_3 \\ \Psi_2 & \Psi_3 & \Psi_4 \end{vmatrix}. \tag{2.13}$$

In fact, the real part of I is $\frac{1}{16}$ times the Kretschmann scalar $R_{\kappa\lambda\mu\nu}R^{\kappa\lambda\mu\nu}$ for the vacuum case.

It may be noted that, for algebraically special space-times, it is necessary that $I^2 = 27J^2$. In this case, if I and J vanish, the space-time must be of types N or III, otherwise is must be of types D or II. Moreover, if $\boldsymbol{k}$ is a repeated principal null direction, a space-time for which I and J are non-zero is of type D if $3\Psi_2\Psi_4 = 2{\Psi_3}^2$, otherwise it is of type II. (For further details see Stephani *et al.*, 2003.) The space-time is also of type D if $\Psi_1 = 0 = \Psi_3$ and $\Psi_0\Psi_4 = 9{\Psi_2}^2$ (see Chandrasekhar and Xanthopoulos, 1986).

In general, including the Ricci scalar R, it is known that there exist 14 real scalar polynomial invariants of the type (2.13), which involve components of both the Ricci and Weyl tensors. However, 14 independent scalars like this are not known explicitly. In practice, it is convenient to define 16 or 17 such scalar quantities and a number of constraints, or syzygies, constraining them. For details of these, their meanings and the relations between them, see Penrose and Rindler (1986), Carminati and McLenaghan (1991), Zakhary and McIntosh (1997) and references contained therein.

2.1.3 Geodesics and geometrical optics

In some local region of space-time, any two events may be joined by a family of curves. Within such a family, the curve which has either maximum or minimum proper length is known as a geodesic. (It is a maximum or minimum according to whether the events have a timelike or spacelike separation, respectively.) For space-times with a metric connection, a geodesic also has the property that its tangent vector is parallelly transported along it (i.e. it is autoparallel). These are two distinct properties of what is intuitively required to generalise the concept of a straight line in flat space to a curved space-time.

Consider a three-parameter family of curves in a region of space-time such that exactly one curve passes through each point. The equations of such a *congruence* can be written in terms of the local coordinates x^μ in the form

$$x^\mu = x^\mu(y^i, s),$$

where y^i, $(i = 1, 2, 3)$ are the parameters identifying particular curves of the congruence and s is a parameter along each. The corresponding vector field $\boldsymbol{v}$ that is tangent to the congruence at all points is given by

$$v^\mu = \frac{\mathrm{d}x^\mu}{\mathrm{d}s}.$$

The congruence is null if $v_\mu v^\mu = 0$, and consists of geodesics if the tangent vectors are parallelly transported along it, i.e. if

$$\frac{D\, v^\mu}{\mathrm{d}s} \equiv v^\mu{}_{;\nu}\, v^\nu = \lambda\, v^\mu,$$

for some $\lambda(y^i, s)$, where $D/\mathrm{d}s$ denotes the derivative along the congruence. The parameter s is called *affine* if $\lambda = 0$, in which case s is defined up to a linear transformation. For such an affine parameter, the above geodesic