Introduction

The purpose of the present book is to give an idea about fundamental concepts and methods, as well as instructive special results, of a unified *intermediate asymptotic* mathematical theory of the flow, deformation and fracture of real fluids and deformable solids. This theory is based on a quite definite and, we emphasize, idealized approach where the real materials are replaced by a continuous medium; therefore it is often called the mechanics of continua. It generalizes, and represents from a unified viewpoint, more focused disciplines: fluid dynamics, gas dynamics, the theory of elasticity, the theory of plasticity etc. For various reasons these disciplines underwent separate development for a long time. The splendid exceptions found in the work of A. L. Cauchy, C. L. M. H. Navier and A. Barré de Saint-Venant in the nineteenth century and L. Prandtl, Th. von Kármán, and G. I. Taylor in the twentieth century confirm rather than disprove the general rule. Therefore the teaching of the mechanics of continua and, more generally, the maintaining of interest in the mechanics of continua as a unified scientific discipline was, in this period of fragmentation, the job of physicists, who considered it to be a necessary part of a complete course of theoretical physics. So it was not by accident that among those who created courses in mechanics of continua were outstanding physicists: M. Planck, A. Sommerfeld, V. A. Fock, Ya. I. Frenkel, L. D. Landau and E. M. Lifshitz and, more recently, L. M. Brekhovskikh.

The physicists were more interested, however, in general ideas and methods rather than in the consistent presentation of special, even very important, results, which in fact give true shape to the subject.

In recent decades the tendency to take the separate branches of the mechanics of continua and combine them into a unified scientific discipline has penetrated to the mechanics community, and the practitioners of various specialities and scientific profiles have been involved in this activity.

There have been several reasons for this tendency; first of all, the internal need for the development of the mechanics of continua in order for one to understand

and evaluate, from a unified viewpoint, the methods and results achieved in the various specialized branches of this subject. Bearing in mind the modern grand scale of activity in mechanics stimulated by practical applications, it was simply not expedient to develop repeatedly from scratch essentially the same ideas and overcome similar difficulties in applications to different branches of the mechanics of continua.

An instructive example: the investigation of the slow motion of viscous fluids in wedge-shaped vessels and the investigation of the stress–strain state in elastic wedges can be reduced to completely identical mathematical problems. Nevertheless, for more than 60 years the study of these two problems was performed separately, without even a single cross-reference, in spite of the now obvious fact that instead of solving from scratch a problem in fluid dynamics it is enough to reinterpret appropriately the solution to the corresponding elasticity problem obtained long before, and vice versa.

It seems therefore expedient to continue to work to unify the subject as much as possible and in particular to avoid the appearance and propagation of separate terminology in its special branches – these constitute a language barrier which prevents the understanding of problem formulation and solutions by even qualified specialists not belonging to a certain narrow community. Such a unification should aid the discovery and use of results about analogous activities in other branches of the mechanics of continua.

Long ago Lord Rayleigh in Great Britain and L. I. Mandelstam in the Soviet Union developed a remarkable idea of “oscillation mutual aid” – a stimulating exchange of ideas and methods between physicists studying oscillations of different types: mechanical, acoustic, electromagnetic etc. The realization of this idea is an instructive example of a fruitful approach unifying various branches of physics on the basis of a unified method. Another instructive example of such a unified approach is synergetics.

The second, and perhaps the most important, reason for promoting the above-mentioned consolidation has been the growth in the range of the materials used in technology and also in the range of parameters (temperature, pressure etc.) of the working conditions of even classical structural materials such as steel. Thus it has become necessary to study the behavior of structures fabricated from synthetic materials and from more traditional materials under extreme conditions: high temperatures, pressures, strain rates and loading rates etc. For such applications the classical models of materials used in the theory of elasticity, fluid dynamics etc. are insufficient and more general ones are needed.

In more detail, the situation is as follows. Every material at fixed external conditions (temperature, pressure etc.) has a certain characteristic “relaxation time” τ. This is the time during which a deformed body in its fixed shape (Figure 1)

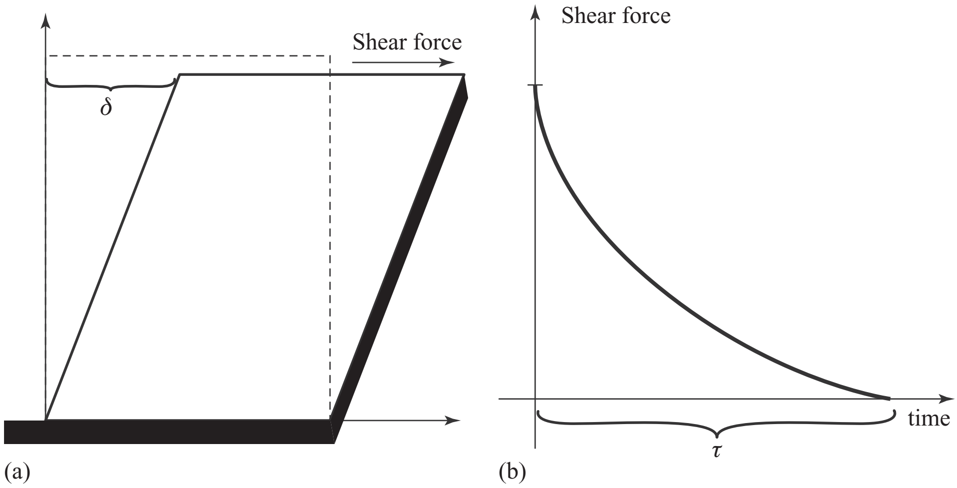

Figure 1 (a) A shear deformation imposed on a body, and the resulting fixed skewed shape of the body. (b) The shear stress disappears after a "relaxation time" τ.

continues to exhibit shear stresses before settling down to a new equilibrium. Every physical process of loading and/or deformation also has its own characteristic time T, e.g. the time duration of application of the load. Therefore the general behavior of a given material in a given loading or deformation process is determined by the ratio of these two times, the dimensionless parameter

$$De = \frac{\tau}{T}, \tag{1}$$

the *Deborah number*, as named by M. Reiner.[1] If $De \ll 1$ then the behavior of the material can be qualified as fluid and if $De \gg 1$ the material can be considered as a solid. So, for water under normal conditions we have $\tau \sim 10^{-12}$ s and for steel (in the range of tensile or compressing stresses of the usual order, 1000 kgf/cm^2) $\tau \sim 10^{12}$ s. The duration T of a human experiment is usually in the range 10^{-9} s (1 nanosecond) $< T < 10^9$ s (30 years). Therefore, for any T in this range, De for water is very small, less than one thousandth, and for steel it is very large, more than one thousand.[2]

[1] The reason for such a name derives from the Song of Deborah in the Old Testament (Judges **5**:5): "The mountains flowed from before the Lord". It is interesting that in the old King James version it was written "melted" instead of "flowed"; careful comparison with the original showed that this was not quite correct (especially for the purpose under discussion) and a proper translation was done in the later Cambridge academic editions of the Bible. The same mistake occurs in the Slavic Church text of the Old Testament.

[2] Now it becomes clear what exactly is the meaning of the statement of Deborah in modern scientific terms: as is known, the characteristic time for rocks (of which "mountains" consist) is of the order of 10^{11} s. However, the characteristic time of the Lord as mentioned by Deborah should be taken, according to modern estimates of the age of the Universe (the time since the Creation (the Big Bang)) to be several billion years, $T > 10^{16}$ s. Therefore, De is less than 10^{-5} and, indeed, rocks ("mountains") can be considered as fluids. This argument

At the same time, for Silly Putty, a well-known material from which a popular toy used to be made, τ is of the order of 1 s. Therefore this material can have both fluid and solid body behavior in an easily accessible range of observation times. Indeed, if it is thrown quickly to the floor it will rebound like a rubber ball; put it on a horizontal solid surface and it will flow slowly along the surface; hammer it sharply and it will break like fragile glass. Similarly, to observe the "solid body" behavior of water, for instance its brittle fracture, and the formation and propagation of cracks, it is necessary to load it with an extremely high strain rate: 10^{14}–$10^{15}\ \mathrm{s}^{-1}$. (Such experiments have been performed!)

For the familiar material polymethylmetacrylate (PMMA), an organic glass, the formation and propagation of cracks can be observed by simply using strain rates of the order of $10^4\ \mathrm{s}^{-1}$, easily accessible with the help of a laser pulse in the regime of free generation. However, for structural materials of a common steel type, at high temperatures or at high stresses, the relaxation time decreases rapidly and then, even at ordinary strain rates and loading rates, steel cannot be considered as a solid.

These reasons have led to an increasing interest in studying and teaching the flow, deformation and fracture of bodies as a unified subject, based on sufficiently general assumptions concerning the material under consideration, and in developing corresponding general approaches.

Scientifically speaking, mechanics is a synthesis of the approaches of physics, applied mathematics[3] and engineering. This is also completely true for the mechanics of continua. However, we will concentrate basically on the applied mathematics and physics view, using engineering ideas only to motivate the formulation and description of problems, and will not address detailed considerations of special engineering problems.

As mentioned above, our plan is to explore the fundamental ideas of the mechanics of continua, to give an idea of its basic methods and approaches as well as its instructive special results. Therefore we will not consider those topics that require for their exposition long calculations, with no compensation provided through the frequent use of the results achieved: such topics should be the subject of special courses and books.

obviously also demonstrates that Deborah was aware of correct estimates of the age of the Universe and, in particular, of the Earth; previous Biblical estimates (several thousand years, $\sim 10^{11}$ s) as well as the estimates of Lord Kelvin (hundreds of thousands of years), which did not take into account the heat generated by radioactivity, would be in conflict with her statement.

[3] There is nowadays considerable discussion concerning the subject of applied mathematics. In fact, its proper understanding is clarified if we remember the famous saying of J. W. Gibbs: "Mathematics is also a language." If so, then on the one hand pure mathematicians can be identified with linguists, who study the laws of the formation, history and evolution of languages. On the other hand applied mathematicians, building models of phenomena, are analogous to writers and poets who use language to create novels, poetry, critical essays etc. Applied mathematics is the art of constructing mathematical models of phenomena in nature, engineering and society.

The subject of fluid mechanics and the mechanics of deformable solids is a classical one. Some of the results we present were obtained many years, even many decades ago. It is well known, however, that each era brings its own perception to classical masterpieces of art: literature, music, paintings relate to what excites the human spirit today. This is equally true for a book on the mechanics of continua: even classical results are understood today quite differently from previous times and often need a completely different interpretation.

It is natural to ask what tools are required in continuum mechanics for modeling the equilibrium, motions, deformation, flow and fracture of real bodies.

First we mention the *general invariance principle*, whose value goes far beyond the borders of mechanics. The invariance principle is formulated in the following way:

> *The laws of equilibrium and motion can be expressed through equations valid for all observers.*

The invariance principle establishes the equivalence, the "possession of equal rights" by all observers, which is physically natural. Its formulation is trivial. We will see, however, that the consequences of this principle can be highly non-trivial.

Furthermore, we have at our disposal *conservation laws*: fundamental laws expressing the conservation (or balance) of mass, momentum, angular momentum, energy etc. These conservation laws give the most important fundamental relations of the mechanics of continua. They are, however, insufficient for obtaining a closed mathematical formulation of a model.

Therefore a very important aspect of any study in the mechanics of continua is to supply the continuous medium under consideration with physical properties. In the classical mechanics of continua this is done through the introduction of *finite constitutive equations*. Such equations are a priori relations that are approximately valid for a certain class of motions of a certain class of materials in which the researcher is interested. They connect the above-mentioned properties of state and/or the motion of real bodies with the internal forces acting in them. As classical examples, we can consider Hooke's law for an ideal elastic body, which establishes the proportionality of stress and strain, or Newton's law for viscous flow, which establishes the proportionality of the stress and the strain rate; see Figure 2. If these equations are applicable and well checked, they allow us in principle to obtain a closed system of equations sufficient to construct a mathematical model. It is tacitly assumed that the parameters entering the constitutive equations, such as the coefficients of proportionality in Hooke's law or in Newton's law mentioned before, are universal constants, valid for the whole class of phenomena under consideration. In fact it means that in these phenomena the microstructure of the material remains the same. There are wide and important classes of

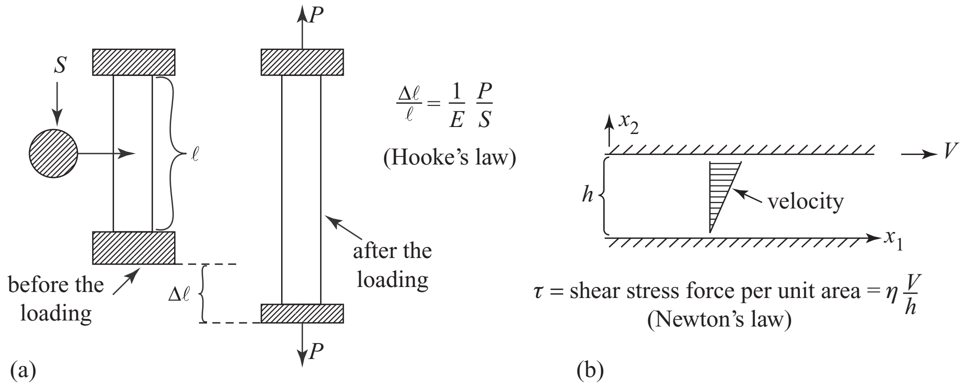

Figure 2 (a) Hooke's law establishes the proportionality of the strain (the elongation per unit length) of a rod and the stress (the tensile force per unit cross-sectional area). (b) Newton's law establishes the proportionality of the strain rate (the velocity difference per unit layer thickness) and the shear force per unit area. The basic content of both laws is that the constants of proportionality η and E are material properties, constant for fixed external conditions.

problems where this is so, and the models provided by the constitutive equations are successful.

The important thing is, however, that in general this is not the case and that therefore the kinetic equations governing the microstructural transformations of the material should also be included in the mathematical model. A classical example is the Zeldovich–Frank–Kamenetsky model of gas combustion, where the laws of mass conservation, energy conservation and the kinetics of the material transformation are considered simultaneously.

Mathematical analysis plays a most important role in the mechanics of continua. In constructing mathematical models of phenomena researchers have created a language of basic concepts that allows our intuition to be shaped. A good mathematical model starts to live on its own, revealing new features of a phenomenon that are very often unexpected for the researcher who created it.

Proving *well-posedness*, i.e. the existence and uniqueness of the solution to a closed system of equations of a mathematical model in a physically reasonable class of functions, under appropriate initial and boundary conditions, is an essential stage of mathematical modeling. However, the problems of the existence and uniqueness of the solution in the mechanics of continua play a much more important role than is generally recognized. There are problems where the really interesting answer is not the solution itself but the range of parameters where solutions exist. Fracture and thermal explosions are classic examples of such problems.

Nevertheless, the central part of the mathematical analysis of problems in the mechanics of continua is not concerned with these topics. An applied mathematician working in this field is interested first of all in obtaining qualitative laws for the phenomena under investigation, i.e. the qualitative properties of the solutions. In a more or less general formulation such qualitative analysis is very difficult although it sometimes happens to be possible. The best chance for investigating the qualitative properties of a phenomenon is given by an explicit analytic form of the solution. To obtain a solution in such a form it is usually necessary to *simplify the problem* formulation as much as possible, in particular to simplify its geometry. Only the most basic features of the phenomenon should be left in the model, those without which the essence of the phenomenon would be lost.[4] It is a very important and responsible stage of modeling. As Einstein is alleged to have said "Things should be as simple as possible, *but no simpler*!" The value of solutions obtained in such a way very often goes far beyond the framework of the corresponding special problems because they can allow clarification of the properties of solutions of wide classes of problems. Textbooks on fluid mechanics, elasticity etc. are not collections of solutions to special problems, as pure mathematicians sometimes claim. Using analytical, and therefore the most transparent, solutions, applied mathematicians working in the mechanics of continua reveal general ideas. Very often special solutions happen to be the intermediate asymptotics to solutions of wide classes of problems where the influence of the non-essential details of the initial and boundary conditions has already disappeared but the system is still far from its state of ultimate equilibrium. Therefore the asymptotic approach pervades continuum mechanics.

For applications it is very important to *compute the solutions* to special problems with sufficient accuracy. Therefore computational methods in mechanics play a most important role. They should not be seen as in opposition to analytical methods. In fact a numerical investigation is especially fruitful after the qualitative features of the expected solutions have been clarified using analytical methods. Numerical computation in general has many features in common with experimental investigation; numerical schemes and algorithms should be carefully investigated and approved in the same way as good experimentalists calibrate their measuring devices. However, the analogy between a numerical computation and an experimental investigation is in fact much deeper than this. The researcher performing a numerical computation cannot be satisfied by a listing of the results of

[4] Compare the poem by Alexander Blok, "Retribution":

Rule out the accidental features
And you will see: the world is marvelous

(Translation from Russian by Sir James Lighthill.)

Figure 3 At some intermediate distance from the picture everyone will recognize the Mona Lisa. Up close the image disappears – it turns out to consist of 560 monochromatic squares distributed in a particular way. However, at large distances the image naturally disappears again. This is an example of an "intermediate asymptotic" consideration, which will be basic in our presentation. From Harmon (1973). Reproduced with permission. Copyright 1973 Scientific American, Inc. All rights reserved.

computations. These results should be processed and attempts made to extract from them some general laws.

A particular example is worthwhile noting: specialists in color printing assign numbers to all colors, for even the finest shades of difference. Therefore the "exact" listing of colors in Figure 3 will constitute a table of the following form:

Square number	Row number	Column number	Number for color of the square
1	1	1	2040–G20Y
2	1	2	4050–G20Y
⋮	⋮	⋮	⋮
560	28	20	2040–G20Y

It is clear that there is more present than just this listing: the Mona Lisa still needs to appear! We will return to this important example later.

The creation of a mathematical model requires the acceptance of certain assumptions, i.e. basic hypotheses. Such an acceptance is always a deliberate act. It should be justified by observation or experiment – a physical or numerical experiment based on a more detailed model. Therefore in the mechanics of continua an appropriate *combination of mathematical modeling with experiment*, be it physical or numerical, is decisive.

The reader may be astonished that the number of tools that have been mentioned is so small. But compare it with the number of tools available to artists: an easel, brushes, palette, canvas, paints – what else . . .? However, the history of civilization would be much poorer without the visual art which has been created using these tools, and only them!

Observation, mathematical modeling and experiment combine to form the basis of continuum mechanics. The goal of this book is to demonstrate how this is done.

1

Idealized continuous media: the basic concepts

1.1 The idealized model of a continuous medium

In the mechanics of continua the most important invention (whose fundamental value, however, is not always appreciated because it seems so natural) is the very concept of a continuous modeling of real materials. More precisely, the truly fundamental discovery was recognizing that to a knowable degree of accuracy the motion, deformation, fracture and/or equilibrium of *real bodies* can be based on an idealization (a *model*), that of a *continuous medium.*

In fact, we intend to study, i.e. to make models of, the motions, deformations, flows, fracture etc. of real bodies. These bodies consist of specific materials: honey, milk, petroleum, metals, polymers, ceramics, rocks, composites etc. If we look at these materials with the naked eye they very often seem continuous and homogeneous. But, when viewed through a microscope or telescope these materials (see Figures 1.1–1.7) display a developed microstructure at various scales – from atomic to essentially macroscopic ones – having a huge diversity of shape. How can we account for this diversity of shapes and properties of the elements of microstructures? Let us forget for the moment that we do not know the equations governing the equilibrium or motion. We do know, however, that taking into account the shape of the elements of a microstructure should mean accepting certain conditions at the boundaries of these odd formations. Let us imagine that by some miracle we know all these odd shapes. It is easy to show that it is impossible to write down the conditions at the boundaries of the microstructural elements even for the simplest problems.

Indeed, consider, for example, the problem of oil flow in an oil deposit. A large oil deposit has a volume of the order of a billion cubic meters. The oil moves in pores whose diameter is of the order of one to ten micrometers. The relative volume occupied by the pores is of the order of one tenth. Therefore in the deposit under consideration there are 10^{23} pores. Let us assume that for a description of a single