The behavior of systems occurring in real life is often modeled by partial differential equations. This book investigates how a user or observer can influence the behavior of such systems mathematically and computationally. A thorough mathematical analysis of controllability problems is combined with a detailed investigation of methods used to solve them numerically, these methods being validated by the results of numerical experiments. In Part I of the book, the authors discuss the mathematics and numerics relating to the controllability of systems modeled by linear and nonlinear diffusion equations; Part II is dedicated to the controllability of vibrating systems, typical ones being those modeled by linear wave equations; finally, Part III covers flow control for systems governed by the Navier–Stokes equations modeling incompressible viscous flow. The book is accessible to graduate students in applied and computational mathematics, engineering, and physics; it will also be of use to more advanced practitioners.
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Exact and Approximate Controllability for
Distributed Parameter Systems
A Numerical Approach

ROLAND GLOWINSKI
University of Houston

JACQUES-LOUIS LIONS
College de France, Paris

JIWEN HE
University of Houston
To Andrée, Angela, and April, and to Dorian Lions
LENS LARQUE-homonyms, with definitions.

1. Lencilorqua: a village of 657 inhabitants on Vasselona Continent, Reis, sixth planet to Gamma Eridani.
3. Laenzle arc: the locus of a point generated by the seventh theorem of triskoïd dynamics, as defined by the mathematician Palo Laenzle (907–1070).


The most challenging course I took in high school was calculus.


The real trick to writing a book is writing. Until you have a book.

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Preface

During ICIAM 1995, in Hamburg, David Tranah approached Jacques-Louis Lions and myself and asked us if we were interested in publishing in book form our two-part article “Exact and approximate controllability for distributed parameter systems” which had appeared in Acta Numerica 1994 and 1995. The length of the article (almost 300 pages) was a justification, among several others, for such an initiative. While I was very enthusiastic about this project, J.L. Lions was more cautious, without being against it. Actually, his reservation concerning this book project was stemming from recent important developments on controllability related issues, justifying, in his opinion an in-depth revision of our article. Both of us being quite busy, the project was practically forgotten. As everyone knows in the Scientific Community, and elsewhere, Jacques-Lions passed away in June 2001, while still active scientifically. He largely contributed in making the Control of Distributed Parameter Systems a most important field where sophisticated mathematical and computational techniques meet with advanced applications. Therefore, when David Tranah renewed his 1995 suggestion during a conference of the European Mathematical Society held in Nice in February 2003, we thought that it would be a very nice way to pay to J.L. Lions the tribute he fully deserves. The idea was to respect as much as possible the original text, since it largely reflects J.L. Lions’ inspired scientific vision, and also its inimitable way at making simple complicated notions. On the other hand, it was also agreed that additional material should be included to make the text more up to date. Most of these additions are concerned with flow control; indeed, for J.L. Lions, the control of flow modeled by the Navier–Stokes equations was a kind of scientific Holy Grail and we are most happy that he could witness the first real mathematical and computational successes in that direction, all taking place in the late 1990s.

The present volume is structured as follows:

- Motivations and some broad generalities are given in the Introduction.
- Part I is dedicated to the control of linear and nonlinear diffusion models; it contains Sections 1–5 of the Acta Numerica article, with additional materials such as the Neumann control of unstable advection–reaction–diffusion models, and a discussion of computer memory saving methods for the solution of time-dependent control problems by adjoint-equation-based methods. A short introduction to Riccati-equation-based control methods is also provided.
Part II is concerned with the controllability of wave equation type models and of coupled systems. This material corresponds essentially to Sections 6 and 7 of the Acta Numerica article.

Part III is the main addition to the original text; it is dedicated to the boundary control, by either rotation or blowing and suction, of Newtonian incompressible viscous flow modeled by the Navier–Stokes equations.

Since most of the additional material follows from investigations conducted jointly with Professor Jiwen He, a former collaborator of J.L. Lions, all the parties involved found it quite natural to have him as a coauthor of this volume.

Acknowledgments and warmest thanks should go first to David Tranah, Ken Blake, and Cambridge University Press for encouraging the publication of this augmented version of the Acta Numerica article, and also to Mrs Andrée Lions and Professor Pierre-Louis Lions for their acceptance of this project. The invaluable help of Dr H.L. Juárez (UAM-Mexico City) and of his collaborators (Bety Arce, in particular) is also acknowledged; they converted large parts of a text initially written in Word© to a LATEX© file, a nontrivial task indeed considering the size of this volume.

Special thanks are due to S. Barck-Holst, M. Berggren, H.Q. Chen, J.M. Coron, J.I. Diaz, S. Gomez, M. Gorman, A.J. Kearsley, B. Mantel, R. Metcalfe, J. Périaux, T.-W. Pan, O. Pironneau, J.-P. Puel, A.M. Ramos, T. Rossi, D. Sorensen, J. Toivanen, and E. Zuazua for very helpful comments and suggestions concerning the additions to the original article (further acknowledgments may be found at the end of this volume; they concern the original Acta Numerica article).

We will conclude this preface with further thanks to Cambridge University Press for authorizing the reprinting of the above Acta Numerica article in Volume III of J.L. Lions, Oeuvres Choisies, SMAI / EDP Sciences, Paris, 2003, a three-volume testimony of the outstanding scientific contributions of Jacques-Louis Lions.

Guanajuato, Mexico

Roland Glowinski